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Physical Entities and Mathematical Representation

Chester H. Page

(August 22, 1961)

Certain basic postulates about physical observables yield the structure of their mathematical representation. Measure equations are contrasted with quantity equations, and measurement units with abstract units. The abstract vector spaces in which observables are represented comprise the core of dimensional analysis.

Systems of equations, units, and dimensions are discussed, along with comments on rationalization. The problem of assigning a dimension to angle is discussed, and a new proposal offered.

Physical systems and experimental situations are quantitatively described in terms of "observables", entities which are subject to measurement. These entities are of various kinds, e.g., force, mass, charge, etc. Two entities are said to be of the same kind if they can be compared, i.e., if the question, $E_1 \leq E_2$?, is meaningful. For example, we can compare one force with another to determine which is larger, but we cannot say whether a given distance is larger or smaller than a particular mass.

The implication of the above remarks is that physical entities can be sorted into ordered chains, or one-dimensional sets, of like entities:

$$q_1 \leq q_2 \leq q_3 \cdots$$

$$p_1 \leq p_2 \leq p_3 \cdots$$

The entities in a chain are said to be of the same "kind"; entities in different chains cannot be compared, and are of different kinds. The common property of entities in the q-chain may conveniently be called "q-ness", for an identifying label.

Observables are not only subject to ordering in sets, but are also subject to measurement. Thus, we can experimentally determine a ratio between two entities in a given chain:

"q_2 is x times as large as q_1".

The experimental procedures of physical science are based on the (arbitrary) choice of a particular sample of an entity as a *measurement unit* for entities of that kind. The general sample of that kind is then described by the statement:

"q is $\{q\}$ times as large as q_u"

where q_u is a symbol for the particular sample chosen to be the measurement unit, and the number, $\{q\}$,[1] is called the value, or measure, of the sample, q. (In contrast, the Moh hardness scale is a familiar example of chain ordering *without* metric properties.)

If the concept of an observable, q, has any "reality", or is to have any quantitative usefulness, the entity q must have a significance that is independent of our choice of measurement unit. The observable must be invariant to our mode of description. If we change our measurement unit from q_{u_1} to q_{u_2}, where

$$q_{u_2} \text{ is } \alpha \text{ times as large as } q_{u_1},$$

then the measure of *any* q, relative to q_{u_2}, must satisfy

$$\{q\}_2 = \{q\}_1/\alpha$$

where $\{q\}_1$ denotes the measure of the same sample of q relative to q_{u_1} as measurement unit.

The experimental laws of physics are expressible as proportions among measures of entities, such as

$$\{q\} = k\{p\}\{s\}/\{r\}.$$

The proportionality constant, k, depends upon the choice of measurement units for the various entities involved, and upon the configuration of the experimental setup. It is customary to think of the measures on the right as relating to more "elementary" entities, whose measurement units have already been chosen. The measurement unit of the "new" entity q is then *defined* by assigning a convenient value to k for a standardized experimental situation.

As we shall see later, a few measurement units can be arbitrarily chosen as a basis for a set of units. These basic units, together with assigned values for the k's in the fundamental equations, determine the remaining measurement units "germane" to the system.

It should be noted that, during the development of a science, the factor k includes the effects of experimental variables that have either inadvertently or intentionally been held constant. For example, the interaction forces among current-carrying conductors depend upon the fluid medium filling the laboratory. The permeability of air varies so little with normal environmental changes, that the effect of this variable would not appear in experiments leading to an initial formulation of Ampere's force law.

[1] Several types of brackets are used for special symbols. A list of these notations is appended to the text.

It should also be noted that there are hidden dependencies among the k's. For example, the interaction force between two stationary charges, the interaction force between two current loops, and the relation between current and charge implied by the magnetic effect of a moving charge, are not independent. The three k's involved satisfy a relation involving the speed of light in the medium in which the experiments are performed. The existence of a relation between a set of k's and the measure of some physical entity implies that at least one k contains a hidden reference to a physical entity not explicitly referred to in the experimental measure equations.

There is a uniqueness postulate in physics, usually not stated, that is relevant to the completeness of a law. "If $\{q\}$ depends upon $\{p\}$, $\{s\}$, etc., then any entity whose measure depends upon the same variables in the same way, and upon no others, is an entity of the same kind as q." Any apparent violation of this postulate is an indication of hidden "variables" in the proportionality constants. These "variables" may be properties of nature which are beyond our power to change, such as the speed of light in vacuo. This constancy prevents the experimenter from including these variables in any particular law, and also allows the theoretician to distribute the effect of hidden variables in any convenient self-consistent manner.

The development of physics as a science, rather than as a taxonomy of empirical relations, was due to the abstract representation of these relations by mathematical relations among mathematical elements, subject to manipulation by algebraic and differential processes. Maxwell wrote:[2]

"There are two methods of interpreting the equations relating to geometry and other concrete sciences.

"We may regard the symbols which occur in the equation as of themselves denoting lines, masses. times, &c.; or we may consider each symbol as denoting only the numerical value of the corresponding quantity, the concrete unit to which it is referred being tacitly understood.

"If we adopt the first method we shall often have difficulty in interpreting terms which make their appearance during the calculations. We shall therefore consider all the written symbols as mere numerical quantities, and therefore subject to all the operations of arithmetic during the process of calculation. But in the original equations and the final equations, in which every term has to be interpreted in a physical sense, we must convert every numerical expression into a concrete quantity by multiplying it by the unit of that kind of quantity."

We next examine the basis of the representation of physical entities (concrete quantities) by mathematical elements, and the structure of the associated mathematics. For simplicity, we shall consider all physical entities as scalars; vectors and tensors can be so described in terms of their components.

Since samples of q can be chain-ordered, these samples can be represented by points in a one-dimensional topological space. In addition, the measurability of observables implies that this is a metric space; and in particular, the measurability as a ratio to a measurement unit implies a one-dimensional abstract vector space. We use the same symbol q to denote both a physical entity and its representative point in the vector space. In its latter role as a mathematical element, the symbolic quantity q is often called a "physical quantity." (This is the reason I have been using the term "entity" for the physical case, to distinguish between an "entity" and its representation by a "quantity".) The duality of the symbol q causes little difficulty; the ambiguity is usually resolved by the context. On the other hand, as we shall see later, substitution of a name for q can cause confusion, such as the use of "ampere" for the name of the measurement unit of current as well as for a mathematical element.

The particular entity sample, q_u, used for a measurement unit, will be represented by a point denoted by $\langle q \rangle$, the base or unit vector of the space in which the q entity is represented. The vector $\langle q \rangle$ is an *abstract* or *symbolic* unit, as contrasted with the more concrete measurement unit. Both abstract and measurement units are often referred to as "units", with no modifier. Thus, to an experimenter, the unit of current is the ampere, and is specified by operational procedures, while to many theorists, the unit of current is also the ampere, but is defined to be a mathematical element!

The relation between an entity sample and its measurement unit, "q is $\{q\}$ times as large as q_u", is represented by the equation

$$q = \{q\}\langle q \rangle.$$

This is an equation relating points or vectors in the space of q-elements. Let $[q]$ be a symbol for this q-space, the representation space for entities of the kind q. We can think of $[q]$ as a symbol for the attribute of "q-ness" of the entity. (Later, $[q]$ will also be called the "dimension" of q.)

Generalization of Measure Equations

If k contains no hidden "variables", the theoretician generalizes the experimenter's measure equation, $\{q\} = k\{p\}\{s\}$, to the abstract relation, $q = kps$, where q, p, and s are now mathematical elements in the $[q]$, $[p]$, and $[s]$ spaces. The innocent looking "product", ps, is not yet defined. However, our postulate on the completeness of a physical law, in combination with the assumed relation between q, p, and s, implies that the product operation has the following property:

"The product of *any* point in $[p]$ and *any* point in $[s]$ is a point in $[q]$, and conversely, any point in $[q]$ can be expressed as the product of some point in $[p]$ and some point in $[s]$." We can therefore define $[q]$ to be the product of the *spaces* $[p]$ and $[s]$:

$$[q] = [p]\,[s].$$

Since the order of changing the p and s conditions in an experiment is immaterial, we can require the

[2] Encyclopaedia Britannica, Ninth Edition, VII, 241.

above multiplication to be commutative:

$$[q]=[p]\,[s]=[s]\,[p]=[ps]=[sp].$$

Note that we are not here defining the result $[q]$ of the product operation, but are defining the product operation by its result.

We can also define the less frequently needed sum of spaces to be a logical sum. Since $[p]$ is the space of p-ness, i.e., the set of all possible values of p (collectively), $[p]+[s]$ can be defined to be the set of all elements possessing p-ness plus the set of all elements possessing s-ness; i.e., the set of all elements possessing either p-ness or s-ness (or both). It follows that

$$[p]+[p]+\ \ldots\ +[p]=[p]$$

$$n[p]=[p]$$

$[n]=[m]=[1]$; the space of numbers, or "numeric space".

Also

$[1]\,[p]=[p]$, since any element of $[1]$ multiplied by any element of $[p]$ yields an element in $[p]$.

The self-consistency of the experimental proportionalities of physics implies a unique multiplication table for the various $[q]$, and in fact that the $[q]$ form a group, with $[1]$ as its identity element.

In general, k may involve hidden "variables". The mathematical model must have internal consistency, and to be useful, should satisfy a *correspondence principle:* The measure equation deduced from a symbolic equation is to be identical with the experimental measure equation being modeled. The measure equation, $\{q\}=k\{p\}\{s\}$, is generalized to

$$q=\langle\Gamma\rangle kps$$

where $\langle\Gamma\rangle$ is a unit vector in an arbitrary space. Since $q\equiv\{q\}\langle q\rangle$, etc., we immediately find the *unit equation*,

$$\langle q\rangle=\langle\Gamma\rangle\langle p\rangle\langle s\rangle,$$

and the space relation,

$$[q]=[\Gamma][p][s]$$

The logical necessity for introducing various $[\Gamma]$ spaces was not widely recognized until the MKSA system of units became popular. The dimensional analysis procedures commonly used by MKSA advocates brought to light the hidden "variables" in the conventional proportionality constants.

A constraint on k's, such as $k_1k_2=\{r\}$, is generalized to

$$k_1\langle\Gamma_1\rangle k_2\langle\Gamma_2\rangle=r=[r]\langle r\rangle,$$

so that

$$\langle\Gamma_1\rangle\langle\Gamma_2\rangle=\langle r\rangle$$

is the corresponding constraint on the assignment of both the spaces and the unit vectors of the Γ's that are introduced for completeness of the theory.

A set of symbolic units which makes the theoretical and experimental measure equations identical is called a "coherent" set. The absence of coefficients in the resulting unit equations shows that the coherent units form a group, with $\langle 1\rangle$ as the identity. $\langle 1\rangle$ is unity, considered as the unit vector of the space of numerics, $[1]$. This makes the set of symbolic units "coherent" with the set of corresponding spaces. A set of germane measurement units is represented by a set of coherent symbolic units.

When we treat physics from the symbolic, or quantity calculus, point of view, we consider the quantity equations as basic and the measure equations as derived. The quantity equations are therefore invariant to choice of units. Thus

$$\{F\}_1\langle F\rangle_1=\{m\}_2\langle m\rangle_2\{a\}_3\langle a\rangle_3$$

even if the systems 1, 2, and 3, are different. For example:

> (Measure of force in newtons) times (newton) equals (measure of mass in pounds) times (pound) times (measure of acceleration in centimeters per second) times (centimeter) divided by (second squared).

This equation *cannot* be factored into

$$\{F\}_1=\{m\}_2\{a\}_3$$

$$\langle F\rangle_1=\langle m\rangle_2\langle a\rangle_3$$

If $\{m\}_2=\alpha\{m\}_1$ and $\{a\}_3=\beta\{a\}_1$, we have

$$\{F\}_1\langle F\rangle_1=\alpha\beta\{m\}_1\{a\}_1\langle m\rangle_2\langle a\rangle_3$$

and the conventional measure equation, $\{F\}_1=\{m\}_1\{a\}_1$, yields

$$\langle F\rangle_1=\alpha\beta\langle m\rangle_2\langle a\rangle_3$$

In fact, the invariance of the symbolic quantities m and a yields directly:

$$\langle m\rangle_1=\alpha\langle m\rangle_2 \qquad \text{and} \qquad \langle a\rangle_1=\beta\langle a\rangle_3$$

so that $\langle F\rangle_1=\langle m\rangle_1\langle a\rangle_1$ is a coherent unit equation.

The above noncoherent units yield the measure equation

$$\{F\}_1=\{m\}_2\{a\}_3/\alpha\beta.$$

In textbooks, the braces and subscripts are omitted and replaced by marginal notes:

"$E=-10^{-8}d\phi/dt$; E in volts, ϕ in maxwells."

In any coherent set of units, the equation has the simple form

$$E=-d\phi/dt.$$

To summarize, the product equations show the following:

(1) $q=kps$ defines the product of p and s

(2) $[q]=[p]\,[s]$ yields the multiplication table of the group of spaces or dimensions

(3) $\langle q\rangle=\langle p\rangle\,\langle s\rangle$, or its equivalent $\{q\}=k\{p\}\{s\}$, yields the multiplication table for the group of coherent units.

Proportionality Constants

We have seen that the theoretician must use $k\langle\Gamma\rangle$, or an equivalent, where the experimenter uses a numeric k. The conventional way of including the hidden "variable" is to make k a nonnumeric. This is done by "correcting" the experimenter's equation to

$$\{q\}=\{k\}\{p\}\{s\}$$

and using the theoretical equations:

$$q=kps$$

$$\langle q\rangle=\langle k\rangle\langle p\rangle\langle s\rangle.$$

A given experimental law, such as Coulomb's Law for electric charges,

$$\{F\}\propto\{q_1\}\{q_2\}/\{r^2\}$$

is frequently subjected to different choices of proportionality constant by different writers. To correlate all these choices, and make them yield the same meaning, requires interpreting them as the measure equations relevant to a *single* symbolic equation, with different choices of units. Thus

$$F=kq_1q_2/r^2$$

yields

$$\{F\}\{r^2\}=\{k\}_i\{q_1\}_i\{q_2\}_i$$

and

$$\langle F\rangle\langle r\rangle^2=\langle k\rangle_i\langle q\rangle_i^2$$

in the ith system of coherent units. We assume a common choice of units for F and r. Since the left hand sides of these equations are independent of i, the choice of unit system, the right hand sides must also be invariant with the choice. Therefore,

$$\frac{\{q\}_a}{\{q\}_b}=\sqrt{\frac{\{k\}_b}{\{k\}_a}},$$

which agrees with the direct interpretation, by an experimenter, of the change of proportionality constant. But we also have $k=\{k\}_i\langle k\rangle_i$, which is invariant, so that the change of the unit of q implies a change of the (symbolic) unit of k. (Since $\langle k\rangle$ was introduced for theoretical convenience, we do not insist on the existence of a measurement unit, k_u, corresponding to the symbolic unit $\langle k\rangle$.)

We can interpret the above as follows. The germane measurement unit of an entity (q) depends upon the choice of proportionality constant ($\{k\}$) in the defining equation for the germane unit. If the symbolic quantity q is to have the same meaning in all cases, there must be a corresponding change in the unit of k, i.e., in the unit of some real or fictitious entity upon which k depends.

It has earlier been mentioned that the experimental proportionality constant, $\{k\}$, depends upon the *geometrical configuration* of the defining experiment. When k contains no hidden "variables" in the usual physical sense, we can interpret it to contain a *geometrical quantity*, represented by some space $[k]$. The change of $\langle k\rangle$ associated with a change of $\{k\}$ now implies a change of some *geometric unit*. For example, the magnetizing force at the center of a current circle is

$$H=kI/r$$

with the rationalized and nonrationalized measures

$$\{k\}_r=1/2$$

$$\{k\}_n=2\pi.$$

If H and k are to have fixed meanings,

$$\{H\}_n\langle H\rangle_n=\{H\}_r\langle H\rangle_r$$

$$\{k\}_n\langle k\rangle_n=\{k\}_r\langle k\rangle_r$$

yielding

$$\frac{\langle k\rangle_r}{\langle k\rangle_n}=\frac{\{k\}_n}{\{k\}_r}=4\pi.$$

Coherence demands

$$\langle H\rangle_n=\langle k\rangle_n\langle I\rangle/\langle r\rangle$$

$$\langle H\rangle_r=\langle k\rangle_r\langle I\rangle/\langle r\rangle$$

so that

$$\frac{\langle H\rangle_r}{\langle H\rangle_n}=\frac{\{H\}_n}{\{H\}_r}=\frac{\langle k\rangle_r}{\langle k\rangle_n}=4\pi.$$

The relation $\langle k\rangle_r=4\pi\langle k\rangle_n$ suggests that k is a solid angle, measured in terms of spheres or steradians as units.

Perucca [3,4] emphasized that the factor 4π appearing in nonrationalized equations of electromagnetism is *not* a numeric, but the quantity 4π steradians. Young [5] introduced a proportionality constant, S, representing the *measure* of a sphere, into various equations.

This interpretation of rationalization is based on a *fixed* symbolic equation, with the measures expressed relative to rationalized or nonrationalized *units*. If, on the other hand, we consider rationalization to be a *change* of the symbolic equation,

$$H_n=2\pi I/r$$

$$H_r=I/2r$$

we have

$$H_n=4\pi H_r$$

so that H_n and H_r do not represent the same identical entity. If the factor 4π is interpreted as a geometrical quantity, H_n and H_r are not strictly of the same kind (dimension). We can still satisfy the experimenter, and avoid paradoxes, by *arbitrarily* setting

$$\{H_n\}=\{H\}_n,\ \{H_r\}=\{H\}_r$$

so that the experimenter need not distinguish between his rationalized measure of H and the measure of a new "rationalized H". The problem will be discussed in detail in a later section.

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[3] E. Perucca, Fusione dei sistemi MKSA razionalizzato e non razionalizzato, French version in La Ricerca Scientifica, p. 1931 (1959).

[4] E. Perucca, Solid angle in electromagnetism, rationalization, calculus of quantities, La Ricerca Scientifica **30,** Suppl. 12, p. 2300/11 (1960).

[5] Leo Young, Electrical units and dimensions, Trans. AIEE **75,** p. 767/71 (1957).

Units and Dimensions

At any given stage in our knowledge of physics, we have n independent proportionalities involving N kinds of entity. The n proportionalities imply constants $k_1 \ldots k_n$.

We can choose $k_1 \ldots k_n$ as arbitrary numbers, and N-n arbitrary measurement units (N-n prototype standards). These N-n units are called "basic", for they serve as a basis of a unit system whose other germane units are derived from these N-n. One of the problems of a standards laboratory, top echelon, is to build experimental set-ups for the realization of the derived units.

We are apt to think of the meter, kilogram, second, degree Kelvin, and candela as the only units defined by prototype standards. The measurement unit for angle is, however, determined by a prototype standard: the circle. Surveyors define the circle to be a realization of 360 units of angle; scientists define it to be a realization of 2π units. Similarly, solid angle can be measured in spheres, steradians, or solid degrees.

Note that we made N arbitrary choices in all; n k's and N-n prototypes. These N choices can be split up differently. For example, we could choose prototype standards for the units of current, voltage, and resistance, and determine the proportionality constant in Ohm's Law experimentally.

We not only have N arbitrary choices for our system of measurement units, but also N choices of representation spaces. If the $k_1 \ldots k_n$ are chosen to be numeric, then we have n unit equations, dimensional equations, or product relations, among N spaces. Hence, N-n spaces can be assigned arbitrary names or dimensional symbols, and the dimensions of the remaining n spaces expressed as products (including reciprocals) of the independent N-n. The N-n independent dimensions become the generators of the group of dimensions, and are also often called "basic". ("Basic", throughout, refers to being a "base" rather than implying the possession of any "fundamental" nature.)

The quantity equation appropriate to a given problem is deduced by application of the n basic laws, hence is a combination of the n basic equations. The answer space can readily be computed as an appropriate product of generators. This is the common algebra-checking application of dimensional analysis.

Choosing all k's to be numeric often discards information. The q-ness, or kind, of a quantity uniquely determines its representation space, but if too many spaces are all called numeric, the converse is no longer true, and a dimensional label does not uniquely identify the kind of quantity. For example, torque and energy are usually given a common dimension label, with a trivial attempt at distinction by discriminating between "foot pounds" and "pound feet". This is esthetically distasteful, since torque is work per unit angle. The confusion arises from assigning $[\theta]=[1]$, either for "convenience" or because Maxwell (loc. cit.) referred to "all quantities essentially numerical, such as exponents and exponentials, logarithms, angles, and circular and elliptic functions . . ."; Maxwell, however, also said (loc. cit.) "By knowing the dimensions of any quantity we are able at once to deduce its numerical value as expressed in terms of one system of units from its numerical value as given in terms of another system."

Langhaar [6] has paraphrased this: "*They are a code for telling us how the numerical value of a quantity changes when the basic units of measurement are subjected to prescribed changes.*" And later (p. 6)

"The unit of temperature may be assigned independently of the units of the entities of mechanics. Consequently, the symbol $[T]$,[7] denoting temperature, is regarded as one of the fundamental dimensions. The dimension of an arbitrary variable z is denoted by $[z]$. . . ."

Since the measure of an angle depends upon the choice of unit (e.g., degree or radian), and the choice of unit is independent of the choices made for the other basic units, we conclude that "angle" is not "essentially numerical", but that the dimensions $[\theta]$ and $[\Omega]$ of plane angle and solid angle are ordinarily suppressed or ignored.

In appendix 2, it is shown that $[\theta]^2=[1]$, $[\sin\theta]=[\theta]$, $[\cos\theta]=[1]$, so that $[\theta]=[1]$ is an allowable assignment, but one which is unduly restrictive. More generally, $[\theta]$ is a square root of [1], not further describable. This is analogous to $j^2=-1$ in complex numbers, and $i^2=j^2=k^2=-1$ in quaternions. The Gibbs' scalar and vector products carry dimensional implications

$$[\cdot]=[\cos\theta]=[1]$$

$$[\times]=[\sin\theta]=[\theta]$$

so that

$$W=\mathbf{F}\cdot\mathbf{d} \text{ yields } [W]=[F][d]$$

while

$$\mathbf{T}=\mathbf{d}\times\mathbf{F} \text{ yields } [T]=[d][\theta][F]$$

so that

$$[T]=[W][\theta]$$

and since $[\theta]^2=[1]$, $[T]=[W]/[\theta]$

which agrees with our verbal description of torque as work per unit angle.

After we assign $\{k_1\} \ldots \{k_n\}$, determining the germane measurement units, we still have arbitrary choices for $[k_1] \ldots [k_n]$. We can make these explicitly, or implicitly by an arbitrary choice of $[q]$ for the "new" quantity in an equation.

For example, the force between two current elements in vacuo is conventionally considered to be [8, 9]

$$d\mathbf{F}=\frac{\Gamma_m}{4\pi} I_1 I_2 \frac{d\mathbf{s}_2\times(d\mathbf{s}_1\times\mathbf{r})}{r^3}$$

The (arbitrary) choice of $\{\Gamma_m\}=4\pi\ 10^{-7}$ defines the measurement unit of current in terms of those of force and distance. Classically, $[\Gamma_m]$ is assigned to be [1] in the electromagnetic system of units and

[6] Henry L. Langhaar, Dimensional Analysis and Theory of Models, p. 5 (John Wiley & Sons, 1959).

[7] The reference uses θ for temperature; I have changed this to avoid confusion with angle in this discussion.

[8] "Conventionally" because it violates Newton's Third Law when applied to incomplete circuits, but not when applied to complete circuits.

[9] The symbol, Γ_m, is the "magnetic constant", often denoted by μ_0.

dimensions, yielding $[I]=[m]^{1/2}\,[l]^{1/2}\,[t]^{-1}$. But if we do not assign a space to Γ_m, we can assign an arbitrary space to I, yielding:[10]

$$[\Gamma_m]=\frac{[F]}{[I]^2}=\frac{[m]\,[l]}{[I]^2[t]^2}$$

This is the customary procedure associated with the MKSA system of units. There are four basic *symbolic* units (meter, kilogram, second, ampere) associated with four basic dimensions, or generator spaces, but there are still only three basic *measurement* units. The measurement unit of current (ampere) is defined as a derived unit by assigning a measure to Γ_m. This illustrates the confusion arising from the use of the word "unit" without a modifier; the (measurement) unit of current is a derived unit, but the (symbolic) unit of current is a basic unit. Even worse is the substitution in the above of the name of a unit. The ampere (measurement unit) is derived; the ampere (symbolic unit) is basic!

Any proportionality constant appropriate to an experiment in vacuo can be thought of as a property of space. This interpretation is frequently employed when the phenomenon involves "action at a distance" and the proportionality constant depends upon the intervening medium. Thus, in Ampere's force law, the proportionality constant is customarily factored into two terms, $\mu\Gamma_m$, where μ is a numeric (unity for vacuum) representing the differences among media. By virtue of a field interpretation of action at a distance, μ is thought of as a measure of the permeability of the medium to the field. Hence, we find the usage:

μ is relative permeability
$\mu\Gamma_m$ is "absolute" permeability
Γ_m is the permeability of space.

The term "permeability of space" implies a property of nature, experimentally determinable, but Γ_m is arbitrarily assignable! The measure, $\{\Gamma_m\}$, *defines* the measurement unit of current. The only property of space involved is represented by the "hidden variable", speed of light, which occurs in the constraining relation between two otherwise arbitrary proportionality factors:[11] $c^2\Gamma_e\Gamma_m=1$.

It is just as logical to interpret the constant, R, in the ideal gas law,

$$pV=RT$$

as a property of space. Elementary thermodynamics shows us that for any ideal gas, $R=C_p-C_v$, the difference between the specific heat at constant pressure and that at constant volume. Experimentally, R is determined by a sequence of measurements of pV/T at lower and lower gas densities, finally extrapolated to zero density! Does this not yield (C_p-C_v) of a vacuum?

[10] If we consider the 4π in the denominator to be a solid angle,

$$[\Gamma_m]=\frac{[F]\,[\Omega]}{[I]^2}.$$

[11] The electric constant, Γ_e, is often denoted by ϵ_0.

The constant $\{R\}$ could have been arbitrarily assigned, thereby defining the measurement unit of temperature. For reasons of convenience, however, the unit of temperature was chosen as basic (determined by a prototype standard), leaving $\{R\}$ as a constant to be measured.

Rationalization

Coulomb's Law is not a good starting point for a discussion of rationalization, despite the intriguing occurrence of $4\pi r^2$. The introduction of 4π into the denominator is compensated by changes of Γ_e and Γ_m to keep force unaffected.

We start instead with the electromagnetic equations relevant to the proportionalities:

$$\oint \mathbf{H}\cdot d\boldsymbol{\lambda} \propto I$$

$$\oint \mathbf{D}\cdot d\boldsymbol{\sigma} \propto q.$$

In particular, the magnetostatic field of a current loop can be expressed in terms of a scalar potential:

$$\phi \propto I\Omega$$

where Ω is the solid angle subtended, at the point of interest, by the current loop. It is convenient to write

$$\phi=-kI\,\frac{\Omega}{S}$$

and

$$\mathbf{H}=kI\,\frac{\nabla\Omega}{S} \qquad (1)$$

where S represents the complete solid angle surrounding a point, i.e., the solid angle subtended by any closed surface at any interior point.

The multivalued nature of the solid angle Ω, along any contour linking the loop, yields

$$\oint \nabla\Omega\cdot d\boldsymbol{\lambda}=S$$

and therefore

$$\oint \mathbf{H}\cdot d\boldsymbol{\lambda}=kI. \qquad (2)$$

We also have the geometric relation (see appendix 2),

$$\{d\Omega\} \propto \{dA\}/\{r^2\}$$

which implies the quantity equation

$$d\Omega=\Gamma_\Omega dA/r^2$$

without prejudice as to the unit or dimension of Ω. Hence, the angle subtended by the loop is

$$\Omega=\Gamma_\Omega\int\frac{\mathbf{r}\cdot\mathbf{n}da}{r^3}$$

leading to

$$\nabla\Omega=\Gamma_\Omega\oint\frac{\mathbf{r}\times d\mathbf{l}}{r^3},$$

and

$$\mathbf{H}=\frac{kI\Gamma_\Omega}{S}\oint\frac{\mathbf{r}\times d\mathbf{l}}{r^3}. \tag{3}$$

The customary rationalized equations of the form (1), (2), and (3) require, respectively·

$$\left\{\frac{k}{S}\right\}=1/4\pi$$
$$\{k\}=1$$
$$\left\{\frac{k\Gamma_\Omega}{S}\right\}=1/4\pi$$

so that

$$\{k_r\}=1$$
$$\{S_r\}=4\pi$$
$$\{\Gamma_\Omega\}_r=1.$$

The customary nonrationalized equations require

$$\left\{\frac{k}{S}\right\}=1$$
$$\{k\}=4\pi$$
$$\left\{\frac{k\Gamma_\Omega}{S}\right\}=1$$

so that

$$\{k_n\}=4\pi$$
$$\{S_n\}=4\pi$$
$$\{\Gamma_\Omega\}_n=1.$$

Thus in *both* cases, solid angle is measured in steradians: $\{S\}=4\pi$, implies $S=4\pi$ (str), and Γ_Ω is a unit vector for accommodating dimensions.

As pointed out by Perucca [3,4], the most suitable interpretation of $\{k_n\}=4\pi$ is $k=4\pi$ (str)$=S$, or (by eq (2)),

$$\oint\mathbf{H}_n\cdot d\boldsymbol{\lambda}=SI$$

If we could interpret $\{k_r\}=1$ as $\{S\}_{\text{spheres}}$, then we could identify H_r with H_n, using a change of solid angle unit to explain the relation $\{H_n\}=4\pi\{H_r\}$. This is Young's interpretation [5] of the effect of rationalization on the coherent unit of solid angle, and would resolve the argument as to whether quantities or units are changed by rationalization,[12] in favor of the "practical" viewpoint. Unfortunately for this viewpoint, we have already found that $\{S_r\}=4\pi$ for consistency. If the rationalized form of (2), $\oint\mathbf{H}_r\cdot d\boldsymbol{\lambda}=I$, is to be taken as a quantity equation, $[k]=[1]$, and (1) becomes

$$\mathbf{H}_r=\frac{I\nabla\Omega}{S}=\frac{I\nabla\Omega}{4\pi\,(\text{str})}$$

again yielding the Perucca interpretation of the 4π factor.

Comparing the rationalized and nonrationalized quantity equations, we see that

$$H_n=SH_r=4\pi\,(\text{str})\,H_r$$

is the *quantity* change interpretation of rationalization. Since $\langle\Omega\rangle=(\text{str})$ in *both* systems, there is no ambiguity in taking measures of this relation, finding

$$\{H_n\}=4\pi\{H_r\}.$$

In both systems, eq (3) requires insertion of the unit vector Γ_Ω as a factor, if this equation is to be considered a quantity equation. There are three choices:

(a) Consider both forms of eq (3) as *measure* equations only,

(b) Insert Γ_Ω explicitly in writing these equations, or

(c) Require $[\Gamma_\Omega]=[1]$.

Choice (c) makes $[\Omega]=[A]/[l]^2=[\theta]$, so that solid angle and plane angle have the same dimension and unit, i.e., solid angle is measured in ordinary radians, or else solid radians and plane radians are both numeric.

The parallel analysis of

$$\nabla\cdot\mathbf{D}=k\rho$$
$$\int\nabla\cdot\mathbf{D}\,d\tau=kq=\oint\mathbf{D}\cdot d\boldsymbol{\sigma}$$
$$\mathbf{D}=\frac{k\Gamma_\Omega}{S}\int\rho\nabla\left(\frac{1}{r}\right)d\tau$$

yields the same conclusions throughout, and the results are compatible with the Maxwell equation for $\nabla\times\mathbf{H}$.

The dimensional conclusions are:

$$[H_r]=[I]/[l]$$
$$[H_n]=[\Omega]\,[I]/[l]$$
$$[D_r]=[q]/[A]=[q]/[\theta]\,[l]^2$$
$$[D_n]=[\Omega]\,[q]/[\theta]\,[l]^2=[\Gamma_\Omega]\,[q]/[l]^2$$

I wish to thank F. B. Silsbee for many stimulating and valuable discussions.

Notation

q	A mathematico-physical quantity, or the corresponding physical entity.
$\{q\}$	The numerical measure of q. If a *particular* unit system is involved, it is indicated by a subscript outside the bracket.
$[q]$	The abstract vector space of the quantity q; also, the "dimension" of q.
$\langle q\rangle$	The element of $[q]$ chosen as a unit vector.
(name of unit)	A particular symbolic unit chosen for $\langle q\rangle$, e.g. (radian).
q_u	The measurement unit of the entity q.

[3,4] See footnotes on p. 230.

[12] F. B. Silsbee, Does rationalization change units?, Elec. Eng. **76**, 296/9 (1957).

Appendix 1

The meaning of $\langle p\rangle\langle s\rangle$ cannot be directly interpreted in terms of experimental conditions.

Consider

$$E=(1/2)mv^2$$

$$\langle E\rangle=\langle m\rangle\langle v\rangle^2$$

The unit equation does *not* imply that a unit mass moving at unit velocity possesses unit energy. Realizations of measurement units are inferred from *measure* equations. A system possesses or exhibits the measurement unit of energy, E_u, when $\{E\}=1$. Now

$$\{E\}=\{m\}\{v\}^2/2$$

so that E_u is realized by a unit mass moving at a speed which is $\sqrt{2}$ times as big as v_u, or by a mass twice as big as m_u, moving at v_u.

Appendix 2

Let us choose the radian as the coherent unit of angle, so that the familiar measure equations

$$\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\cdots$$

$$\sin\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\cdots$$

can be adopted, without change, as quantity equations. Then

$$\cos\theta=1-\frac{\{\theta\}^2\langle\theta\rangle^2}{2!}+\cdots$$

$$\sin\theta=\{\theta\}\langle\theta\rangle-\frac{\{\theta\}^3\langle\theta\rangle^3}{3!}+\cdots$$

$$=\{\theta\}\langle\theta\rangle\left(1-\frac{\{\theta\}^2\langle\theta\rangle^2}{3!}+\cdots\right)$$

For each right hand side to represent an element in a single space, rather than in a mixture of spaces, we must have

$$[\theta]^2=[1].$$

This implies that any variable x such that its space $[x]$ satisfies $[x]^2=[1]$ is an allowable argument for the trigonometric functions. Thus, sin $\{\theta\}$ is also defined by the infinite series.

Since $\langle\theta\rangle_{\text{coherent}}=$(radian), (radian)$^2=\langle 1\rangle$. If we choose a noncoherent unit, say,

$$\langle\theta\rangle=\alpha\ (\text{radian})^{13}$$

then

$$\langle\theta\rangle^2=\alpha^2$$

and

$$\alpha\{\theta\}=\{\theta\}_{\text{rad}}$$

This yields

$$\sin\theta=\frac{\langle\theta\rangle}{\alpha}\left(\alpha\{\theta\}-\frac{\alpha^3\{\theta\}^3}{3!}+\cdots\right)$$

$$\equiv\frac{\langle\theta\rangle}{\alpha}\sin\alpha\{\theta\}=\frac{\langle\theta\rangle}{\alpha}\sin\{\theta\}_{\text{rad}}$$

so that

$$\{\sin\theta\}\langle\sin\rangle=\langle\theta\rangle\sin\{\theta\}_{\text{rad}}/\alpha=(\text{radian})\sin\{\theta\}_{\text{rad}}$$

where $\langle\sin\rangle$ is the unit vector in the space where sines are represented.

It follows that $[\sin]=[\theta]$, and if a given space is to have a unique unit vector, $\langle\sin\rangle=\langle\theta\rangle$. For coherence, the symbolic unit of sine is the radian.

The consequences of $[\theta]\neq[1]$ are interesting. The vector area of a parallelogram is $\mathbf{r}_1\times\mathbf{r}_2$, and the volume of a parallelopiped is $\mathbf{r}_1\times\mathbf{r}_2\cdot\mathbf{r}_3$. These yield

$$[\text{Area}]=[l]^2[\theta]$$

$$[\text{Volume}]=[l]^3[\theta]$$

which appear odd, but are compatible with the Gauss Divergence Theorem:

$$\int\nabla\cdot\mathbf{V}\,d\tau=\oint\mathbf{V}\cdot d\boldsymbol{\sigma}$$

and Stokes' Theorem:

$$\int\nabla\times\mathbf{V}\cdot d\boldsymbol{\sigma}=\oint\mathbf{V}\cdot d\boldsymbol{\lambda}$$

Solid angle is often defined in terms of spherical area by

$$d\Omega=dA/r^2$$

or its equivalent

$$d\Omega=\sin\theta\,d\theta\,d\phi$$

With $[\text{Area}]=[l]^2[\theta]\cdot$ and $[\sin\theta]=[\theta]$, $[\theta]^2=[1]$, these formulations agree dimensionally and yield $\langle\Omega\rangle=\langle\theta\rangle=$(radian). The lack of distinction between "solid angle" and "plane angle" is analogous to the lack of distinction between "plane angle" and "numeric" when $d\theta=dl/r$ is used as a quantity equation.

We can avoid all reference to area by considering a finite acute solid angle bounded by planes (fig. 1). At each edge, the dihedral angle can be expressed in terms of its supplement, τ_i, the "turning angle" of the normal to the edge. It is readily shown (in terms of the excess of a spherical triangle) that the measure of the solid angle is proportional to the amount by which $\sum_i\tau_i$ fails to be a complete turn:

$$\{\Omega\}_{\text{hemispheres}}=\{\text{U.P.T.}\}_{\text{circles}}$$

where U.P.T. is the "uncompleted part of one turn".

The general convention for relating solid angle units to the corresponding plane angle units is that the measure of a hemisphere in solid units is the same

[13] "(radian)" is not a parenthetical reference to units, but the mathematical element "radian".

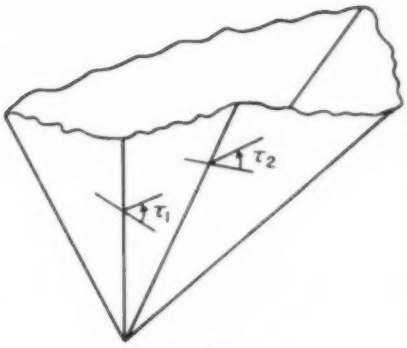

FIGURE 1

as the measure of a circle in plane units, e.g., a hemisphere is 2π steradians or 360 solid degrees. Thus for *any* unit of plane angle, we define the germane unit of solid angle through

$$\{\Omega\}_{\text{solid }\theta}=\{\text{U.P.T.}\}_{\text{plane }\theta}$$

This implies

$$\frac{\Omega}{\langle\Omega\rangle_i}=\frac{\text{U.P.T.}}{\langle\theta\rangle_i}$$

And therefore

$$\frac{\langle\Omega\rangle_i}{\langle\theta\rangle_i}=\frac{\Omega}{\text{U.P.T.}}$$

But the right hand side is invariant to unit choice, therefore $\langle\Omega\rangle_i=\Gamma_\Omega\langle\theta\rangle_i$ with Γ_Ω invariant to unit choice. For coherence, $\{\Gamma_\Omega\}=1$, and Γ_Ω becomes a unit vector whose only role is to distinguish between solid angle and plane angle.

As shown before, the usual rationalized electromagnetic equations are conveniently interpreted by letting $\Gamma_\Omega\equiv 1$, with no dimensional distinction between solid angle and plane angle. This feature is built into the mathematics by using $d\Omega=dA/r^2$.

The geometrical definition of sine and cosine as length ratios associated with a right triangle offers a dimensional paradox. It is, however, easily resolved by realizing that we are dealing with a special case of the law of sines:

$$\frac{l_1}{\sin\theta_1}=\frac{l_2}{\sin\theta_2}=\frac{l_3}{\sin\theta_3}$$

This equation places no restriction on [sin]. The paradox comes from setting $\sin\pi/2=\langle 1\rangle$, rather than $\sin\pi/2=\langle\sin\rangle=\langle\theta\rangle$.

It is also interesting to note that $e^{j\theta}$ is not only two-dimensional in the sense of having both real and imaginary parts, but is also two-dimensional as having components in two spaces:

$$e^{j\theta}=\cos\theta+j\sin\theta$$

$$[\cos\theta]=[1]$$

$$[\sin\theta]=[\theta].$$

This is an illustration of the fact that equations used in physics need not be dimensionally homogeneous, when terms arise from the use of an artifice. The form $e^{j\theta}$ does not actually occur in physics; it is introduced with the convention that its real part alone (or its imaginary part alone) represents a variable of interest. Nonartificial exponentials, such as $e^{-w/kT}$, have numeric exponents.

The turning-angle approach to solid angle leads to a formula for the solid angle subtended at a point by an arbitrary closed space curve:

$$\Omega=2\pi-\oint r\frac{\mathbf{r}\times\mathbf{r}'\cdot\mathbf{r}''}{(\mathbf{r}\times\mathbf{r}')\cdot(\mathbf{r}\times\mathbf{r}')}\,ds$$

where $r'=d\mathbf{r}/ds$, and ds is any parameter of progression around the curve.

Appendix 3

It is interesting to note the relation between "turns" and "radians". Although one complete turn is equivalent to 2π radians, i.e., any closed plane curve subtends 2π radians at any internal point, the concept of "turn" is topological and not metric. The number of *complete* passes around *any* closed path is independent of our concept of angle. But as soon as we try to measure fractional turns, we must define a metric concept for the interpolation and this involves defining a unit for measuring the angle of a partial turn. We must also specify the location of the point around which we are considering the partial turn.

Consider a circular path around a long straight line, as in the circuital law derivation of the magnetic field near a long straight current-carrying conductor. This turn "obviously" involves 2π radians. If, however, we consider a nonplanar closed curve, we realize that there are points on the wire for which $\oint d\theta>2\pi$; the plane angle concept is not really applicable.

In electromagnetic theory, *complete* encirclements (linkage) and enclosures are involved in the basic concepts and equations. Resolution of total effects into partial effects associated with plane angles and solid angles leads to paradoxes, as in the conventional force law for current elements.

If "turn" is topological and not metric, it should not appear as a dimension or unit. Thus "ampere-turn per meter" is a hybrid term; "ampere per meter" should suffice. The inclusion of "turn" in this unit for H is also redundant, since linkage relations and current paths cannot have fractional turns; any linkage of a path with χ ampere-turns is simply a linkage with χ amperes.

(Paper 65B4–60)

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On the Range of a Fleet of Aircraft

A. J. Goldman

(June 21, 1961)

J. N. Franklin has determined the maximum range of the last member of a fleet of n aircraft with identical fuel capacities and gallons-per-mile fuel efficiencies, on the assumptions that in-air exchange of fuel is unrestricted and that aircraft can be abandoned from the fleet at any stage. The present paper extends the solution (a) to the case of aircraft with equal fuel efficiencies but different fuel capacities, and (b) to the case of aircraft with equal fuel capacities but different fuel efficiencies.

1. Introduction

In a previous paper,* J. N. Franklin considers the problem of determining the maximum range attainable by a fleet of n aircraft with an initial total supply of g gallons of fuel. It is assumed that the ith aircraft has a fuel capacity of g_i gallons and a fuel efficiency of r_i gallons per mile. The aircraft may share fuel in flight, and any of them may be abandoned at any stage.

Franklin gives an explicit solution for $n=2$, which illustrates a dynamic programing method available for larger n and also indicates the complexity of the solution in the general case. He also gives a complete discussion (including an asymptotic formula for g as a function of the maximum range) of the *symmetric case* in which

$$g_i=G, \quad r_i=R \qquad \text{for } i=1, 2, \ldots, n.$$

In the present paper, Franklin's solution for the symmetric case is extended (a) to the situation in which the aircraft of the fleet all have the same fuel efficiency, though their fuel capacities may differ, and (b) to the case of equal fuel capacities with (possibly) different fuel efficiencies.

2. Equal Fuel Efficiencies

In this section it will be assumed that

$$r_i=R \qquad \text{for } i=1, 2, \ldots, n \tag{1}$$

and that the aircraft are so numbered that

$$0<g_1\leq g_2 \ldots \leq g_{n-1}\leq g_n. \tag{2}$$

It is convenient to define

$$G_0=0; \quad G_k=\sum_{j=1}^{k} g_{n+1-j} \qquad \text{for } k=1, 2, \ldots, n; \quad G_{n+1}=\infty. \tag{3}$$

The maximum range will be denoted by $M_n(g)$, or more fully by

$$M(g; g_1, \ldots, g_n)=M_n(g). \tag{4}$$

It will be shown that, if k is the unique integer defined by

$$G_k<g\leq G_{k+1}, \tag{5}$$

then the maximum range is given by

$$M_n(g)=R^{-1}\left\{\frac{g}{k+1}+\frac{G_1}{1\cdot 2}+\frac{G_2}{2\cdot 3}+\cdots+\frac{G_k}{k(k+1)}\right\} \qquad \text{if } g\leq G_n, \tag{6}$$

$$M_n(g)=R^{-1}\left\{\frac{G_n}{n}+\frac{G_1}{1\cdot 2}+\frac{G_2}{2\cdot 3}+\cdots+\frac{G_{n-1}}{(n-1)n}\right\} \qquad \text{if } g>G_n. \tag{7}$$

In the optimal policy, if $g\leq G_n$, then the flight is begun with aircraft $n, n-1, \ldots, n-k$. If $g>G_n$, then $g-G_n$ gallons of fuel must be discarded and the flight is begun with all n aircraft. In any case, if the flight is begun with aircraft $n, n-1, \ldots, n-t$, then the last-listed of these aircraft is abandoned when only G_t gallons of fuel are left, the next-to-last is abandoned when only G_{t-1} gallons remain, etc.

This result will be proved by induction on n. If $n=1$, then formulas (6) and (7) become

$$M_1(g)=\min\,(g/R, g_1/R),$$

which is obviously true. Now assume the result holds for all fleets of $n-1$ aircraft, where $n>1$. Let $x\geq 0$ be the distance at which the first abandonment of an aircraft occurs. (Setting $x=0$ indicates that one or more aircraft are left at the base.) Then

$$M_n(g)=\max_{x\leq g/nR}\{x+\max_{1\leq j\leq n} M(g-nRx; g_1, \ldots, \hat{g}_j, \ldots, g_n)\} \tag{8}$$

*J. N. Franklin, The range of a fleet of aircraft, J. Soc. Ind. Appl. Math. 8 (1960), 541–548.

for $g \leq G_n$, where the caret denotes an omitted argument. The inductive hypothesis shows that

$$M(y; y_1, y_2, \ldots, y_{n-1})$$

is a non-decreasing function of its last $n-1$ arguments over the set

$$0 \leq y_1 \leq y_2 \leq \ldots \leq y_{n-1},$$

and so, using (2), (8) becomes

$$M_n(g) = \max_{x \leq g/nR} \{x + M(g - nRx; g_2, \ldots, g_n)\} \quad (9)$$

for $g \leq G_n$. The inductive hypothesis shows that the polygonal function

$$P(x) = x + M(g - nRx; g_2, \ldots, g_n) \quad (10)$$

of x, at all points in $0 < g - nRx < G_{n-1}$ for which it has a derivative, has a *negative* derivative, and so $P(x)$ is decreasing for $0 < g - nRx < G_{n-1}$.

If $g \leq G_{n-1}$ then the maximum of $P(x)$ required in (9) is therefore obtained by setting $x = 0$, yielding

$$M_n(g) = M(g; g_2, \ldots, g_n). \quad (11)$$

But by the inductive hypothesis, the right side of (11) is given by the right side of (6) for $g \leq G_{n-1}$. Thus (6) is proved for $g \leq G_{n-1}$.

Next suppose that $G_{n-1} < g \leq G_n$. The polygonal function $P(x)$ is decreasing for $0 \leq g - nRx \leq G_{n-1}$, as noted above, and is *increasing* for $g - nRx > G_{n-1}$, since by (7) and the inductive hypothesis the second summand in (10) is constant in this range. Thus the maximum of $P(x)$ required in (9) is obtained by setting $g - nRx = G_{n-1}$, and (9) becomes

$$M_n(g) = \frac{g - G_{n-1}}{nR} + M(G_{n-1}; g_2, \ldots, g_n) = \frac{g - G_{n-1}}{nR}$$

$$+ R^{-1} \left\{ \frac{G_{n-1}}{n-1} + \frac{G_1}{1 \cdot 2} + \ldots + \frac{G_{n-2}}{(n-2)(n-1)} \right\}$$

$$= R^{-1} \left\{ \frac{g}{n} + \frac{G_1}{1 \cdot 2} + \ldots \frac{G_{n-1}}{(n-1)n} \right\},$$

so that (6) is proved for $G_{n-1} < g \leq G_n$. Finally, (7) follows from (6) and the obvious fact that

$$M_n(g) = M_n(G_n) \qquad \text{if } g > G_n.$$

3. Equal Fuel Capacities

In this section it will be assumed that

$$g_i = G \qquad \text{for } i = 1, 2, \ldots, n \quad (12)$$

and that the aircraft are so numbered that

$$r_1 \geq r_2 \ldots \geq r_{n-1} \geq r_n > 0. \quad (13)$$

It is convenient to define

$$R_k = 1/\sum_{j=1}^{k} r_{n+1-j} \qquad \text{for } k = 1, 2, \ldots, n. \quad (14)$$

The maximum range is denoted $M_n(g)$, or more fully by

$$M(g; r_1, \ldots, r_n) = M_n(g). \quad (15)$$

If $g \leq nG$, then there is a unique integer k for which

$$kG < g \leq (k+1)G. \quad (16)$$

The maximum range is given by

$$M_n(g) = G(R_1 + R_2 + \ldots + R_k) + R_{k+1}(g - kG) \qquad \text{if } g \leq nG, \quad (17)$$

$$M_n(g) = G(R_1 + R_2 + \ldots + R_n) \qquad \text{if } g > nG. \quad (18)$$

In the optimal policy, if $g \leq nG$, the flight is begun with aircraft $n, n-1, \ldots, n-k$. If $g > nG$, then $g - nG$ gallons of fuel must be discarded and the flight is begun with all n aircraft. In any case, if the flight is begun with aircraft $n, n-1, \ldots, n-t$, then the last-listed of these aircraft is abandoned when only tG gallons of fuel are left, the next-to-last is abandoned when only $(t-1)G$ gallons remain, etc.

The proof of this result is just like that in the preceding section, and is therefore omitted. By way of orientation, it may be noted that the analog of (9) is

$$M_n(g) = \max_{x \leq R_n g} \{x + M(g - R_n^{-1}x; r_2, \ldots, r_n)\}.$$

(Paper 65B4–61)

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Measurement of Wave Fronts Without a Reference Standard:

Part 1. The Wave-Front-Shearing Interferometer

James B. Saunders

(June 26, 1961)

The wave-front-shearing interferometer may be used to test any converging wave front regardless of whether or not it is symmetrical. A mathematical operation is described that permits complete analysis of the data. This operation yields values of the deviations of wave fronts under test from a close fitting sphere. The reference surface may be chosen statistically so that the results are the deviations from a best fitting sphere.

Introduction

The wave-front-shearing interferometer [1, 2, 3] [1] provides a comparison of a converging wave front with a sheared image of itself. Similarly, the wave-front-reversing interferometer [4] provides a comparison of either a converging or plane wave front with a reversed image of itself. A detailed description of a method of analyzing the data from these interferometers, except for the special case of revolution symmetry, has not been published. Consequently, the absolute shape of a completely unknown wave front could not be obtained.

The method of analysis involves a mathematical operation. This operation is similar for the two aforementioned interferometers. Its description is more clearly portrayed in its application to the wave-front-shearing interferometer (≡WSI) than to the wave-front-reversing interferometer (≡WRI), although the method evolved from a study of the latter. There are, however, sufficient differences in the two operations to warrant separate treatments. Accordingly, part 1, which follows, will describe the analysis of data from a WSI and part 2, to be published later, will describe the analysis of data from a WRI. Part 3 of this series will describe the absolute testing of optical flats. This method is not new but has not been described adequately in the literature.

Part 1

1. Analysis of WSI Fringes

The WSI will yield unique solutions of a wave front. The assumption of symmetry (i.e., that it have revolution symmetry) is unnecessary. Any convergent wave front may be tested along any chosen diameter or along the line of its intersection with any plane that passes through the point of convergence. Several equally spaced reference points are chosen along this line. The line of reference points must be parallel to the direction of shear and the shear must be an integral multiple of the separation of the points. This arrangement and spacing of the reference points causes their images, in the region of overlapping wave fronts, to coincide in pairs as is shown in figure 1A.

The separation of the reference points will be defined as unity for this discussion. If the magnitude of the shear is unity also, one adjustment of the system (one set of fringes) is sufficient to evaluate the deviations of the wave front at the chosen reference points from a statistically chosen sphere. For a shear that exceeds unity, more than one fringe pattern (i.e., two or more shear values) must be used to obtain a solution.

In figure 1, $P_\nu(\nu=0, 1, 2 \ldots N)$ represents the $(N+1)$ reference points in one of the images, W, of the sheared wave front and P'_ν the corresponding points in the other image, W'. Let δ_ν (see fig. 1B) equal to the deviation of the wave front at P_ν, from the corresponding point, T_ν, on a reference circle, C, (to be chosen later), ϵ equals the angle between the two images, C and C', of the reference circle at their point of intersection and μ equals the distance from the intersection point to P_0. The distance, μ, is positive if P_0 is below (in fig. 1B) the intersection of the two circles and negative if above. The deviation, δ_ν, is positive if P_ν is on the concave side of the circle, and negative if it lies on the convex side of the reference circle. The distance from P_0 to P_ν (or from T_0 to T_ν), measured along the circle, is always positive and is represented by ν.

The separation, S_ν (distance from T_ν to $T'_{\nu-1}$), of the two images of the reference circle at any pair of points, P_ν and $P'_{\nu-1}$, (assuming unit shear) may be obtained from figure 2. The centers of the two circles are located at C_1 and C'_1. The distance $C_1C'_1$ equals $2h$, R equals the radii of the two circles, ρ equals the distance from E to T_ν, E bisects the line $C_1C'_1$ and ϕ is the angle subtended by $(\nu+\mu)$ at E. Applying the law of cosines (from trigonometry) to triangles $EC'_1T'_{\nu-1}$ and EC_1T_ν we obtain

$$R^2=h^2+(\rho+S_\nu)^2-2h(\rho+S_\nu)\cos(90-\phi) \text{ and}$$

$$R^2=h^2+\rho^2-2h\rho\cos(90+\phi).$$

[1] Figures in brackets indicate the literature references at the end of this paper.

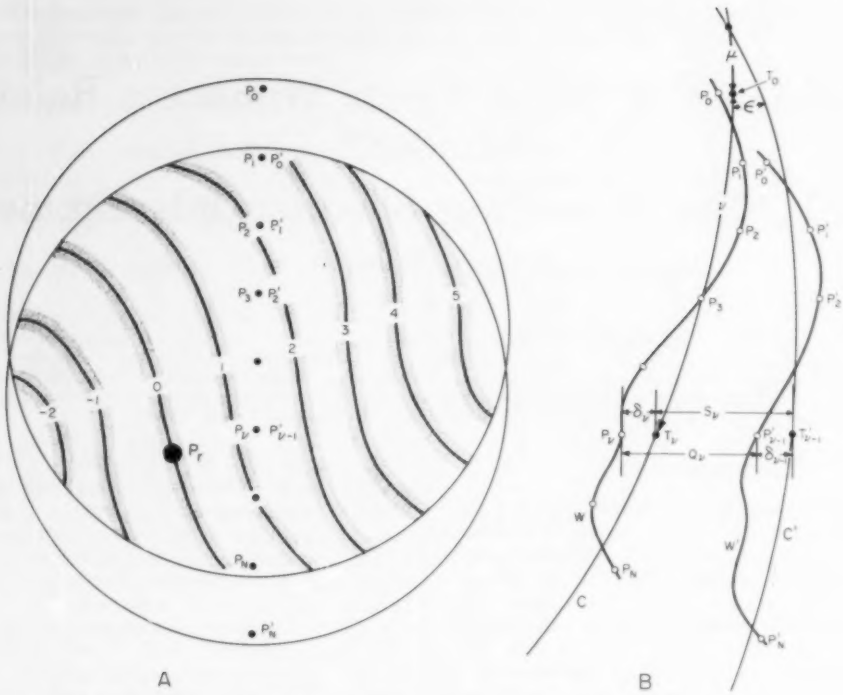

FIGURE 1A *Two images of a wave front sheared laterally relative to itself.*
B *Illustration of the two wave fronts relative to the images of a reference circle.*

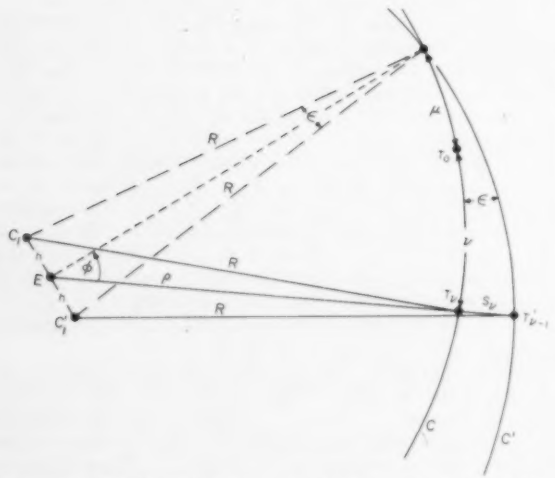

FIGURE 2 *Relationship between parameters and other variables.*

On taking differences and solving for S_ν we obtain,

$$S_\nu = 2h \sin \phi. \tag{1}$$

This equation may be applied rigorously, but if the focal ratio (radius of curvature of the circle divided by the diameter of the wave front) exceeds six, no significant error is introduced by replacing $\sin \phi$ by its approximate equivalent, $(\nu+\mu)/R$. Also, since the angle ϵ will always be quite small, the value of $2h$ is approximately equal to $R \cdot \epsilon$ and the value of S_ν in eq (1) becomes

$$S_\nu = (\nu+\mu)\epsilon. \tag{2}$$

If we let Q_ν represent the separation of the two wave fronts, W and W', at P_ν, it is seen from figure 1B that

$$\left.\begin{aligned} &\delta_\nu = \delta_{\nu-1} + Q_\nu - S_\nu = \delta_{\nu-1} + Q_\nu - \epsilon(\nu+\mu) \\ &\nu = 1,\ 2,\ 3\ \ldots\ N. \end{aligned}\right\} \tag{3}$$

All quantities that represent optical distances are given in units of the wavelength that is used. Accordingly, these quantities should be multiplied by the wavelength to convert to standard units of length. The quantity Q_ν may be observed directly if white light is used to adjust the zero order to a known point, P_r, and then, using monochromatic light of known wavelength, observe the order difference between points P_r and P_ν. It is inconvenient and unnecessary to use white light. Let Q_r represent the unknown order of interference at an arbitrarily chosen point, P_r, in the fringe pattern (fig. 1A) and q_ν the difference in order (number of fringes) between this point and P_ν. The order of interference, Q_ν, at

P_ν is, therefore, equal to $(Q_\nu+q_\nu)$. This introduces another unknown into the equations of observation; but this may be absorbed by replacing the product $\epsilon\mu$ with a new parameter, r, such that

$$\epsilon\mu=(Q_\nu-r). \tag{4}$$

With these values for Q_ν and $\epsilon\mu$ replaced by their respective equals, eq (3) becomes

$$\left.\begin{aligned}&\delta_\nu=\delta_{\nu-1}+q_\nu+r-\nu\epsilon\\&\nu=1, 2\ \ldots\ N\end{aligned}\right\} \tag{5}$$

Equation (5) represents N equations that contain $(N+3)$ unknowns: namely, $(N+1)$ δ_ν's plus the two parameters ϵ and r. We need three additional equations relating these $(N+3)$ unknowns if a solution is to be possible. It appeared that one could use a second set of fringes formed with a shear of two units, as shown in figure 3A. This would yield $(N-1)$ additional equations of observations with only two new unknowns, i.e., two more parameters, ϵ_2 and r_2, to be added to the above-mentioned $(N+3)$ unknowns. If N is 6 or larger, we would have as many equations as unknowns and a solution would seem to be possible, but it can be shown that these equations are not entirely independent.

The δ_ν's do not have significance until the reference circle, C, shown in figure 1B, is defined. Any circle may be defined by three conditions. In analytical geometry, these might be the coordinates of the center and the radius of the circle. A circle might also be defined as the one that passes through three given points. The circle of reference may be fully defined by three equations of condition. When these are combined with the set of observation equations (eq (5)) data from a single set of fringes is sufficient for a complete solution.

The values of the deviations may be obtained without knowing the value of R. However, when testing a parabolic mirror, for instance, the deviations represent the difference between the parabola and a sphere. If comparison is to be made between the measured deviations of the wave front and the computed deviations of a parabola from a circle, the radius of curvature of the circle must be known. The radius, R, can be measured independently of the interferometer. The ideal image point, E, (fig. 2) or the mean point of convergence of the wave front, can be located and its distance from any chosen point can be measured directly. The center of the reference circle will be located at the ideal image point if the reference circle is chosen as the circle that best fits the observed wave front. This may be done most precisely by the method of least squares. The value of R is the measured distance from the ideal image point (point of convergence) to any chosen reference point plus the computed deviation of this reference point from the reference circle.

In the test of a concave mirror, R is approximately equal to the radius of the mirror. In the test of a lens (simple or compound system), R is the distance from the point of convergence (image point) to the back surface of the lens.

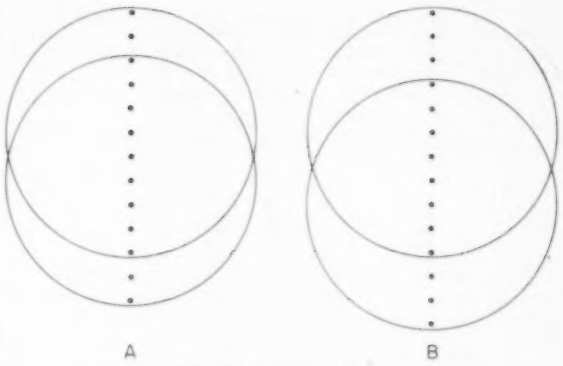

FIGURE 3A *Wave fronts sheared 2 units.*
B *Wave fronts sheared 3 units.*

2. Specifications for the Reference Circle

There are three relatively simple methods of defining the circle of reference. Each method requires three equations of condition. These conditions involve three well-known principles: namely, the method of coincidence, the method of averages, and the method of least squares.

2.1. The Method of Coincidence

We may define the reference circle, shown in figures 1B and 2, by requiring it to pass through any three of the chosen reference points. Thus, we might require that it pass through the two end points, P_0 and P_N, plus some other point near the axis of the mirror (or lens). If the points, P_0 and P_N, are too close to a dubbed edge, a better fitting circle is obtained by requiring it to pass through the two points adjacent to P_0 and P_N, respectively. These two choices for reference circle are made by equating the corresponding δ_ν's to zero. Any three of the points can be chosen to define the circle and the results would be the deviations of the wave front at all chosen reference points from the chosen circle.

For illustration we will choose the fringe pattern shown in figure 1A. There are 8 reference points. We will arbitrarily require the reference circle to pass through points P_0, P_3, and P_7. This is done by requiring that

$$\delta_0=\delta_3=\delta_7=0. \tag{6}$$

For this illustration, eq (5) represents 7 equations of observation. These 7 equations, when combined with the 3 condition equations, (6), form a set of 10 linear equations with 10 unknowns; and all δ_ν's that are not evaluated in eq (6) can be computed in terms of known quantities (the q_ν's and ν's). Table 1 and figure 4 show the results obtained by this method.

2.2. The Method of Averages

The method of averages will usually yield a better fitting circle than the method of coincidence, because of the averaging of the δ_ν's. This method requires an assignment of weights. A reasonable correlation

TABLE 1. *Deviations of a wave front from reference circles that are chosen as follows: Column 3, by the method of coincidence; column 4, by the method of averages; and column 5, by the method of least squares*

ν	q_ν	δ_ν		
		MC	MA	LS
0		0.00	−0.20	−0.10
1	2.74	.23	.13	.14
2	2.10	.15	.11	.07
3	1.70	.00	.00	−.07
4	1.45	−.06	−.05	−.13
5	1.30	.06	.04	−.01
6	1.00	.21	.14	.15
7	0.30	.00	−.16	−.06

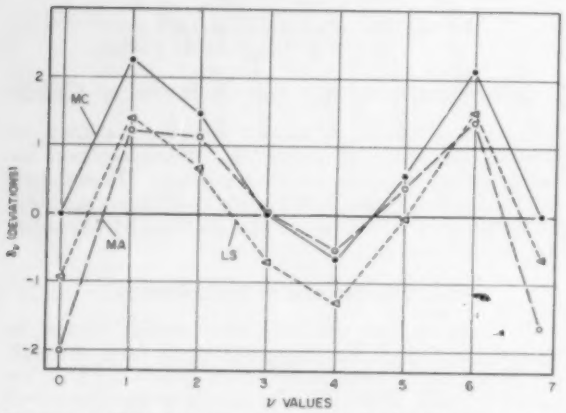

FIGURE 4 *Graphical representation of the deviations of a wave front from reference circles that are chosen by the method of coincidence (MC), the method of averages (MA), and the method of least squares (LS).*

of image quality, formed by the measured wave front from an optical, image-forming system, with wave-front shape would normally require that the weights should be proportional to some function of the distance from the reference points to the center of the circular aperture of the optical system. This would allow the assignment of equal weights to equal areas of the wave front. Equal weights will, however, be used in this discussion.

In previous applications of the method of averages [5], the deviations of the observations from a curve, which is to be fitted to the observations by adjustment of parameters, are equated to functions of the parameters. These are then grouped into as many groups as there are parameters to be adjusted. The algebraic sum of the deviations in each group is equated to zero. When the functions of the parameters (eq (5)) are substituted for the deviations, there are as many equations as parameters and the parameters can be evaluated. Any manner of grouping the deviations will yield a solution. A very poor and sometimes useless solution will be obtained if the points that are allocated to each group are scattered among those of the other two groups. This author prefers to avoid any intermingling of the points of one group with those of other groups.

The parameters of the reference circle shown in figure 1A are ϵ, r and any one of the deviations, δ_ν.

The parameter ϵ (see figs. 1B and 2), together with R, determines the distance, h, from the center, C_1, of the reference circle to the point of convergence, E, of the wave front. The center of this circle is displaced from the ideal image point, E (mean point of convergence) for two reasons. One is that the axis of the mirror or lens cannot be adjusted to pass absolutely through the image of the source, and, secondly, the interference fringes would be too broad for reading when the wave front is nearly spherical. Consequently, the angle ϵ is adjusted to be large enough to provide fringes of a most convenient width for accurate readings.

The parameter, μ, together with R, determines the direction of the point C_1 from E, relative to the axis of the mirror. The parameter r is related to and replaces μ through eq (4). Consequently, the parameters r and ϵ define the center of the circle of reference relative to the wave front.

Any one of the δ_ν's may be chosen as a parameter. By adding δ_ν to the directly measurable distance from E to P_ν we have the radius R of the circle of reference. We will choose δ_0 for the third parameter because of simplification in the formulas that follow.

This relationship of the three parameters to the coordinates of the center of the circle and its radius is somewhat vague, involved, and perhaps not clearly explained, but it is sufficient to understand that they correspond to the three parameters that define the circle of reference.

If equal weights are to be assigned, there must be an equal number of points allocated to each of the three equations of condition. In the example chosen above (fig. 1A), the number of points are not integrally divisible by three. However, we may consider that each of the observation equations represents three identical observations. The total number of observation equations is then $3N$ and division by 3 is now possible. The three equations of condition are:

$$3\delta_0+3\delta_1+2\delta_2=\delta_2+3\delta_3+3\delta_4+\delta_5=2\delta_5+3\delta_6+3\delta_7=0. \quad (7)$$

The set of equations represented by (5) may be replaced by the equivalent set

$$\delta_\nu=\delta_0+\sum_{\sigma=1}^{\nu} q_\sigma+\nu\cdot r-\epsilon\sum_{\sigma=0}^{\nu-1}\sigma=f_\nu \quad (8)$$

where f_ν is a function of the three parameters, δ_0, ϵ, and r. Any equation in set (8) is obtained by adding the first ν equations in set (5). The functions, f_ν, may now be substituted for the corresponding δ_ν's in (7). It will be noted that there is no f_0 corresponding to δ_0. This, however, does not present any difficulty because δ_0 is itself a parameter and may be left in eq (7) as is.

The three equations of (7), after replacing δ_ν by its equal, f_ν, contain the three parameters as unknowns which may be evaluated. The computed values for the parameters are then substituted in eq (8) for evaluating the remaining δ_ν's. The results

for the chosen example, shown in figure 1A, are given in table 1, column MA, and are also represented graphically in figure 4.

The number of reference points, in the unit-shear method, is determined by the shear angle and diameter of the element being tested. For testing elements (mirrors or lenses) during the polishing and figuring operations, the optician is only interested in departures from a desired figure. The easiest method of computing this is, therefore, the most practical. The shear may be adjusted to such a value that the resultant number of points is some multiple of three. The computation is a minimum when the number of points is the same in each group.

2.3. The Method of Least Squares

The method of evaluating a wave front by least squares is more elaborate than the others but would usually be used when highest precision is desired. In applying this method, the sum of the squared deviations $\left(\sum_{\sigma=0}^{N}\delta_\sigma^2\right)$ is to be minimized with respect to the three parameters required to define the reference sphere.

This application of least squares differs from the usual method of least squares in that the observations are assumed to be free from error and the number of observations needs not exceed the number of unknowns whose values are sought (namely the δ's). In fact, a unique solution may be obtained even if the number of observations is one less than the number of chosen reference points.

To minimize the sum of the squared deviations of the wave front from the reference sphere we require that

$$\sum_{\nu=0}^{N}\delta_\nu^2=\delta_0^2+\sum_{\nu=1}^{N}f_\nu^2=\text{minimum}. \tag{9}$$

This is effected by equating to zero the differentials of eq (9) with respect to r, ϵ, and δ_0, respectively. On performing the differentiation and dividing through by 2, we obtain

$$\delta_0+\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial\delta_0}=\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial r}=\sum_{\nu=1}^{N}f_\nu\frac{\partial f_\nu}{\partial\epsilon}=0. \tag{10}$$

The partial differentials, as obtained from eq (8), are

$$\left.\begin{aligned}&\frac{\partial f_\nu}{\partial\delta_0}=1,\ (\nu=0,1,2\ldots N)\\&\frac{\partial f_\nu}{\partial r}=\nu,\text{ and }\frac{\partial f_\nu}{\partial\epsilon}=-\sum_{\sigma=0}^{\nu-1}\sigma,\ (\nu=1,2,3\ldots N)\end{aligned}\right\} \tag{11}$$

On substituting these differentials into (10), we have for the example of 8 reference points (fig. 1A),

$$\left.\begin{aligned}&\delta_0+\delta_1+\delta_2+\delta_3+\delta_4+\delta_5+\delta_6+\delta_7=0\\&\delta_1+2\delta_2+3\delta_3+4\delta_4+5\delta_5+6\delta_6+7\delta_7=0\\&\delta_2+3\delta_3+6\delta_4+10\delta_5+15\delta_6+21\delta_7=0\end{aligned}\right\} \tag{12}$$

On combining sets of eqs (8) and (12), we again have 10 equations with 10 unknowns. Using the same set of fringes (fig. 1A) as was used above for illustration, we obtain for the δ_ν's the values given in column LS of table 1. These values are also plotted, along with those obtained by the other two methods, in figure 4.

3. Sensitivity

The sensitivity of the WSI varies with the absolute magnitude of the shear. If the shear is maintained at unity the sensitivity of the observations increases with a decrease in the number of points because the unit, as defined above, increases. The reduction of the number of points is, of course, limited.

In testing surfaces that depart very greatly from a sphere, as for a large aperture and large aperture-to-focus-ratio parabolic mirror, it is desirable to reduce the sensitivity. Otherwise, such a mirror cannot be tested interferometrically with the WSI. As the number of points is increased, the absolute spacing of the points decreases. The smaller the shear relative to aperture, the smaller can the optical path difference be made. By reducing the absolute magnitude of the shear the maximum optical path difference may be reduced at will and, consequently, the fringes may be adjusted to any desired width.

The sensitivity may also be increased, without reducing the number of reference points, by using two or more shear values that are greater than unity (fig. 3). This may yield more equations than there are unknowns, but averaging of computed points is always acceptable. Each additional set of fringes, obtained with a different shear value, introduces two additional unknowns—new values for ϵ and r.

4. Symmetrical Surfaces

Optical surfaces that are generated mechanically generally have revolution symmetry about the optical axis. If this is known and the axis is known to be at the center of the mirror (assuming circular apertures) then the deviations are known to be symmetrical because the center of the reference circle can be assumed to lie on the axis. Two other conditions are then sufficient to define the reference circle. The condition for revolution symmetry is defined by the specification

$$\delta_\nu=\delta_{N-\nu} \tag{13}$$

and the condition equations, (6), (7) and (12), become, respectively,

$$\delta_0=\delta_3=0, \tag{14}$$

$$3\delta_0+3\delta_1+2\delta_2=6\delta_3+2\delta_2=0, \tag{15}$$

and

$$\left.\begin{aligned}&\delta_0+\delta_1+\delta_2+\delta_3=0\\&21\delta_0+15\delta_1+11\delta_2+9\delta_3=0\end{aligned}\right\}. \tag{16}$$

The third equation in each of the sets (6), (7), and (11) becomes identical, respectively, to the first equation of the set.

The assumption of symmetry (eq (13)) reduces the number of δ_r's by approximately $\frac{1}{2}N$ (exactly $\frac{1}{2}N$ if N is an even number) and reduces the number of condition equations by one. If N is larger than 2, the number of equations exceeds the number of unknowns, for a unit shear arrangement. Consequently, larger shears (2 or more units) may be used with increased sensitivity, as explained above.

5. References

[1] W. J. Bates, A Wavefront Shearing Interferometer, Proc. Phys. Soc. **59,** 940 (1947).

[2] R. L. Drew, A Simplified Shearing Interferometer, Proc. Phys. Soc. (London) sec. B, **64,** 1005 (1951).

[3] Andre Danjon, Mach Interferometer for the Measurement of Fringe Visibility; Arthur Beers, Vistas in Astronomy, p. 383 (Pergamon Press, 1955).

[4] Pol Mollet, Optics in Metrology, The Wavefront Reversing Interferometer, p. 227 (Pergamon Press, 1960).

[5] J. B. Scarborough, Numerical Mathematical Analysis, p. 357 (The Johns Hopkins Press, Baltimore, Md. (1930).

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On the Evaluation of the Function

$$\phi(\lambda)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{u\ln u+\lambda u}du$$

for Real Values of λ*

Wolfgang Börsch-Supan [1]

(June 26, 1961)

The evaluation of the function mentioned in the title is done by deriving asymptotic formulas with error terms for the large values of $|\lambda|$. For values of λ in the remaining finite interval, Simpson's rule is applied after suitable transformation of the path of integration and restriction to a finite portion of this path. A table of the function is given together with guesses of the errors due to the discretization and the finite integration limits.

1. Introduction

In his investigation of the energy loss of fast particles by ionization, Landau [1] [2] uses the function mentioned in the title in order to describe the transformation of the spectral distribution of energy. He gives a short description of this function for its real arguments, including a plot and an asymptotic formula for the function itself and its integral with respect to λ.

For the numerical computation of the energy loss of a certain beam of particles in a certain medium, this function has to be tabulated and/or approximated to a certain accuracy. This requires asymptotic formulas and the corresponding error bounds for large values of λ, and, furthermore, numerical evaluation of the integral for a certain set of finite values after a suitable transformation of the path of integration.

The results are given in the following paragraphs.

2. Transformation of Path of Integration

As pointed out by Landau [1], the evaluation of the integral can be done either directly, by choosing a suitable value σ, or by transforming the path of integration. In the former case the best choice is a line of integration passing through the saddle point of the integrand. The derivative of the integrand vanishes for

$$u=e^{-(1+\lambda)},$$

so the best choice is

$$\sigma=e^{-(1+\lambda)}. \tag{1}$$

The integral can be transformed into a real one by introducing $y=\mathrm{Im}\ \{u\}$ as a new independent variable and by combining the complex conjugate values of the integrand for complex conjugate values of u. Using (1) we obtain the integral representation

$$\phi(\lambda)=\frac{e^{-\sigma}}{\pi}\int_0^\infty \exp\left\{\frac{\sigma}{2}\ln\left(1+\frac{y^2}{\sigma^2}\right)-y\arctan\frac{y}{\sigma}\right\}$$
$$\cdot\cos\left\{y\left[\frac{1}{2}\ln\left(1+\frac{y^2}{\sigma^2}\right)-1\right]+\sigma\arctan\frac{y}{\sigma}\right\}dy. \tag{2}$$

For large positive values of λ this vertical line in the complex plane passes near the singularity of the integrand at the origin. Furthermore, the phase angle of the integrand varies very fast. Therefore this path of integration is not suitable for numerical integration. As the integrand is fast decreasing in the left half-plane for these values of λ, we pull the path of integration from $\sigma\pm i\infty$ to the left to $-\infty$. But the singularity at the origin prevents a further movement of the middle part of the path of integration. Since we have a branch point at the origin, we put a branch cut along the negative real axis and transform the integral into an integral along this branch cut, from $-\infty$ to 0 on the lower side and from 0 to $-\infty$ on the upper side. The integrand differs by the difference of 2π in the imaginary part of $\ln u$. So we get

$$\phi(\lambda)=\frac{1}{2\pi i}\int_{-\infty}^{0}(e^{-\pi i u}-e^{\pi i u})e^{u\ln|u|+\lambda u}du$$
$$=\frac{1}{\pi}\int_0^\infty e^{-v(\ln v+\lambda)}\sin\pi v\,dv. \tag{3}$$

The singularity at the origin is now multiplied by a factor of the order v if compared with the original integrand. Furthermore, the integrand decreases the faster the larger λ is.

It turns out that there is a large range in which both formulas are of practical use, but formula (2) is preferable for large negative λ, and formula (3) for large positive λ. At the same time, these formulae provide asymptotic formulas for $\phi(\lambda)$, given already by Landau, but without any remainder terms or error bounds. We extend Landau's considerations in order to get such error bounds.

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[2] Figures in brackets indicate the literature references at the end of this paper.

3. Asymptotic Formula for Large Negative λ

For large negative λ the asymptotic behavior of $\phi(\lambda)$ is obtained by applying the saddle-point method to eq (2). The integrand can be written in the form

$$\exp\left\{\sigma f\left(\frac{y}{\sigma}\right)\right\}\cos\left\{\sigma g\left(\frac{y}{\sigma}\right)\right\},$$

where

$$f(t)=\tfrac{1}{2}\ln(1+t^2)-t\arctan t,$$

$$g(t)=t(\tfrac{1}{2}\ln(1+t^2)-1)+\arctan t.$$

The following inequalities can be proved by elementary considerations which are too lengthy to be reproduced here in detail:

$$-\frac{t^2}{2}\le f(t)\le -\frac{t^2}{2}+\frac{t^4}{12} \qquad \text{for } t\ge 0, \tag{4a}$$

$$f(t)\le -t\left(\frac{t_0}{2}-\frac{t_0^3}{12}\right) \qquad \text{for } t\ge t_0>0, \tag{4b}$$

$$0\le g(t)\le \frac{t^3}{6} \qquad \text{for } t\ge 0. \tag{4c}$$

Replacing $f(t)$ by $-\frac{t^2}{2}$ and $g(t)$ by 0 yields the asymptotic expression

$$\phi(\lambda)\sim\frac{e^{-\sigma}}{\pi}\int_0^\infty \exp\left(-\frac{y^2}{2\sigma}\right)dy=\frac{\sqrt{\sigma}e^{-\sigma}}{\sqrt{2\pi}}$$

which was given by Landau.

Let us now look for a lower bound of $\phi(\lambda)$ or for a lower bound of the integrand. We replace $f(t)$ by $-t^2/2$, $\cos\sigma g$ by $1-(\sigma g)^2/2$ and then $g(t)$ by $t^3/6$. Integration yields the lower bound

$$\phi(\lambda)\ge\frac{\sqrt{\sigma}e^{-\sigma}}{\sqrt{2\pi}}\left(1-\frac{5}{24\sigma}\right). \tag{5}$$

It is a little more complicated to get an upper bound, since different inequalities are to be used for small t and for large t. We replace $\cos\sigma g$ by 1 for all t, $f(t)$ by the upper bounds given by (4a) for $t\le t_0$ and given by (4b) for $t\ge t_0$, respectively. The quantity t_0 is chosen later, depending upon σ. For the sake of closed integration, the expression $\exp\{t^4/12\}$ is replaced by the upper bound $1+(\exp\{t_0^4/12\}-1)t^4/t_0^4$ for $t\le t_0$. Now the integration can be carried out, provided to $t_0<\sqrt{6}$, and we obtain

$$\phi(\lambda)\le\frac{e^{-\sigma}}{\pi}\int_0^{\sigma t_0}\exp\left\{-\frac{y^2}{2\sigma}\right\}\left(1+[e^{\sigma t_0^4/12}-1]\frac{y^4}{\sigma^4 t_0^4}\right)dy + \frac{\exp\left\{-\sigma t_0\left(\frac{t_0}{2}-\frac{t_0^3}{12}\right)\right\}}{\frac{t_0}{2}-\frac{t_0^3}{12}}$$

Since the integrand of the remaining integral is positive for $y>\sigma t_0$ too, we replace σt_0 by ∞ and obtain

$$\phi(\lambda)\le\frac{\sqrt{\sigma}e^{-\sigma}}{\sqrt{2\pi}}\left(1+\frac{3[e^{\sigma t_0^4/12}-1]}{\sigma^2 t_0^4}+\sqrt{\frac{8}{\pi}}\frac{e^{\sigma t_0^4/12}\cdot e^{-\sigma t_0^2/2}}{\sqrt{\sigma}\,t_0(1-t_0^2/6)}\right).$$

The value of t_0 should be chosen in such a way that the sum of the two last terms is a minimum. But the equation determining this minimum for general σ is rather complicated. So we restrict ourselves to a rather crude discussion and to the case of large σ. The second term in parentheses is minimized for a fixed σ by letting $\sigma t_0^4\to 0$. Then this term goes to $1/(4\sigma)$. But the third term goes to infinity then. Now let us put σt_0^4 equal to some constant, say 12ϵ for convenience. Then our inequality has the form

$$\phi(\lambda)\le\frac{\sqrt{\sigma}e^{-\sigma}}{\sqrt{2\pi}}\left(1+\frac{e^{\epsilon}-1}{\epsilon}\cdot\frac{1}{4\sigma}+\frac{2}{\sqrt{\pi}}\frac{e^{\epsilon}\cdot e^{-\sqrt{3\sigma\epsilon}}}{\sqrt[4]{3\sigma\epsilon}\left(1-\sqrt{\frac{\epsilon}{3\sigma}}\right)}\right), \tag{6}$$

valid for every λ and for every positive $\epsilon<3\sigma$. Apparently, for $\sigma\to\infty$ the last term goes much faster to zero than the middle term. So one can take ϵ small, as long as $\sigma\cdot\epsilon$ remains large. Then the first term approaches $1/(4\sigma)$.

4. Asymptotic Formula for Large Positive λ

The asymptotic behavior of $\phi(\lambda)$ for large positive λ can be seen from (3). In a manner similar to Landau's, we transform the independent variable λ and the integration variable v. The reason is that the interval where the integrand is essentially different from zero and contributes to the integral essentially should be independent of λ for the new integration variable t. Such a transformation is given by

$$\lambda=w+\ln w+B$$

$$v=\frac{t}{w}, \tag{7}$$

where the new variables are w and t, and B is a constant chosen below. So we get

$$\phi(\lambda)=\frac{1}{\pi w}\int_0^\infty e^{-t}e^{-t(\ln t+B)/w}\sin\frac{\pi t}{w}\,dt.$$

Taylor expansion of the second exponential factor yields

$$\phi(\lambda)=\frac{1}{w}\int_0^\infty e^{-t}\sin\frac{\pi t}{w}\,dt-\frac{1}{\pi w^2}\int_0^\infty e^{-t}t(\ln t+B)\sin\frac{\pi t}{w}\,dt + \frac{1}{2\pi w^3}\int_0^\infty e^{-t}t^2(\ln t+B)^2\sin\frac{\pi t}{w}\,e^{-\frac{\theta(t,w)\cdot t(\ln t+B)}{w}}\,dt,$$

where $0<\theta=\theta(t,w)<1$.

In order to obtain a remainder term of the order w^{-4} we would like to choose B in such a way that

the second term becomes zero. But then B would depend upon w. Now the same goal can be reached with a constant B if one develops the sin-term in a Taylor series too:

$$\sin\frac{\pi t}{w}=\frac{\pi t}{w}-\frac{1}{6}\left(\frac{\pi t}{w}\right)^3\cos\frac{\theta_1\pi t}{w}$$

where $0<\theta_1=\theta_1(t,w)<1$. Now we demand only that the part of the second integral corresponding to the first term in the Taylor expansion of $\sin \pi t/w$ disappears, i.e., B is determined by the equation

$$\int_0^\infty e^{-t}t^2(\ln t+B)dt=\Gamma'(3)+B\Gamma(3)=0,$$

which leads to

$$B=C-\frac{3}{2}=-0.922\,784\,335\ldots$$

where C is the Eulerian constant, often also called γ (see [2], especially eq (4) of paragraph 1.1 and (9) of paragraph 1.7). Then it follows that

$$\phi(\lambda)=\frac{1}{w^2+\pi^2}+\frac{\pi^2}{6w^5}\int_0^\infty e^{-t}t^4(\ln t+B)\cos\frac{\theta_1\pi t}{w}\,dt$$

$$+\frac{1}{2\pi w^3}\int_0^\infty e^{-t}t^2(\ln t+B)^2\sin\frac{\pi t}{w}\,e^{-\frac{\theta t(\ln t+B)}{w}}dt. \qquad (7^*)$$

In the first integral we replace the cos-term by the lower bound -1 for $t\le e^{-B}$, and by the upper bound 1 for $t\ge e^{-B}$. The resulting integral can be written as

$$\int_0^\infty e^{-t}t^4(\ln t+B)dt-2\int_0^{e^{-B}}e^{-t}t^4(\ln t+B)dt\le 17.$$

The first term is evaluated in terms of the gamma function and its derivative. The second term is estimated by using $-(\ln t+B)\le t^{-1}$ and by some elementary calculations. These arguments show that the modulus of the integral in the second term of (7*) is smaller than 17.

The third term of (7*) can be treated in a similar way. We replace the sin-term by its bound $\pi t/w$ and the exp-term containing θ by its maximum value $\exp\{w^{-1}\exp(-B-1)\}$. The remaining integral can be expressed in terms of the gamma function and its first and second derivatives (see [2], l.c.).

So finally we have the asymptotic formula

$$\phi(\lambda)=\frac{1}{w^2+\pi^2}+R(w), \qquad (8)$$

where

$$|R(w)|\le\frac{\frac{\pi^2}{2}-\frac{15}{4}}{w^4}\exp\left\{\frac{0.926}{w}\right\}+\frac{17\pi^2}{6w^5}\equiv R_1(w). \qquad (9)$$

and w and λ are connected by (7).

It is more convenient to have the right hand sides of (8) and (9) in terms of λ. Therefore one needs easily computable estimates of w in terms of λ. The uncertainty of w due to this estimation induces an uncertainty in the right hand side of (8), which in turn can be expressed by a corresponding change of λ in the left hand side of (8). Therefore, in addition, we need an estimate of $\phi'(\lambda)$. Using the value

$$w_1=\lambda\left(1-\frac{\ln\lambda+B}{\lambda+1}\right) \qquad (10)$$

as an approximation for w, we obtain the inequalities

$$\frac{1}{w_1^2+\pi^2}-R_1(w_1)$$

$$-\frac{(\ln\lambda+B)^2\exp\{0.926/(w_1-\ln w_1-B)\}}{(w_1-\ln w_1-B)^3w_1^2(1+\lambda^{-1})^3}$$

$$\times(1+w_1^{-1})$$

$$\le\phi(\lambda)\le\frac{1}{w_1^2+\pi^2}+R_1(w_1). \qquad (11)$$

They are valid provided $w_1\ge e^{-}$ $=2.52$.

5. Comments on the Table

The numerical evaluation of (2) and (3) for a given value of λ requires the introduction of finite step lengths and of finite integration limits. As far as finite integration limits are concerned, the errors introduced by this approximation can be estimated rigorously. The errors due to the use of finite step lengths and Simpson's rule are much more involved if expressed in terms of easily computable quantities. Repeated estimates would lead to very poor final estimates. Therefore, a mixed theoretical-experimental way of guessing these errors was adopted. These guesses and some test runs with halved step lengths allow the assumption that the relative error of the values of $\phi(\lambda)$ as given in the table below do not exceed the amounts shown below. Together with the estimates of errors due to finite integration limits, one obtains

$$|\textit{relative error in table}|\le\begin{cases}1.8\times10^{-4} & \text{if } -4\le\lambda\le2\\ 3.2\times10^{-4} & \text{if } 2\le\lambda\le10\\ 7\times10^{-4} & \text{if } 10\le\lambda\end{cases}$$

For values of $\lambda\le-4$ the formulas (5), (6), and (1) give the value of $\phi(\lambda)$ with a relative error not exceeding 1.5 percent. For values of $\lambda\ge100$ the formulas (11), (10), and (9) provide values of $\phi(\lambda)$ with a relative error not exceeding 0.02 percent.

The notation of the numbers is almost obvious: e.g., 1.25 $E-04$ means 1.25×10^{-4}.

λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$
−4. 000	3. 3900E−09	−2. 500	9. 6369E−03	−1. 000	1 5140E−01	0. 500	1. 6524E−01
−3. 975	5. 4974E−09	−2. 475	1. 0633E−02	−0 975	1. 5335E−01	0. 525	1 6433E−01
−3. 950	8. 8066E−09	−2. 450	1. 1700E−02	−0. 950	1. 5523E−01	0. 550	1. 6341E−01
−3. 925	1. 3940E−08	−2. 425	1. 2840E−02	−0. 925	1. 5705E−01	0. 575	1. 6247E−01
−3. 900	2. 1811E−08	−2. 400	1. 4054E−02	−0. 900	1. 5880E−01	0. 600	1. 6152E−01
−3. 875	3. 3740E−08	−2. 375	1. 5344E−02	−0. 875	1. 6048E−01	0. 625	1. 6056E−01
−3. 850	5. 1619E−08	−2. 350	1. 6712E−02	−0. 850	1. 6208E−01	0. 650	1. 5959E−01
−3. 825	7. 8123E−08	−2. 325	1. 8157E−02	−0. 825	1. 6362E−01	0. 675	1. 5861E−01
−3. 800	1. 1700E−07	−2. 300	1. 9682E−02	−0. 800	1. 6509E−01	0. 700	1. 5762E−01
−3. 775	1. 7342E−07	−2. 275	2. 1285E−02	−0. 775	1. 6649E−01	0. 725	1. 5661E−01
−3. 750	2. 5449E−07	−2. 250	2. 2968E−02	−0. 750	1. 6781E−01	0. 750	1. 5561E−01
−3. 725	3. 6984E−07	−2. 225	2. 4731E−02	−0. 725	1. 6907E−01	0. 775	1. 5459E−01
−3. 700	5. 3235E−07	−2. 200	2. 6571E−02	−0. 700	1. 7026E−01	0. 800	1. 5357E−01
−3. 675	7. 5919E−07	−2. 175	2. 8490E−02	−0. 675	1. 7138E−01	0. 825	1. 5254E−01
−3. 650	1. 0729E−06	−2. 150	3. 0486E−02	−0. 650	1. 7243E−01	0. 850	1. 5150E−01
−3. 625	1. 5029E−06	−2. 125	3. 2558E−02	−0. 625	1. 7341E−01	0. 875	1. 5046E−01
−3. 600	2. 0872E−06	−2. 100	3. 4703E−02	−0. 600	1. 7433E−01	0. 900	1. 4942E−01
−3. 575	2. 8743E−06	−2. 075	3. 6921E−02	−0. 575	1. 7518E−01	0. 925	1. 4837E−01
−3. 550	3. 9259E−06	−2. 050	3. 9209E−02	−0. 550	1. 7596E−01	0. 950	1. 4732E−01
−3. 525	5. 3195E−06	−2. 025	4. 1565E−02	−0. 525	1. 7668E−01	0. 975	1. 4627E−01
−3. 500	7. 1518E−06	−2. 000	4. 3985E−02	−0. 500	1. 7734E−01	1. 000	1. 4521E−01
−3. 475	9. 5423E−06	−1. 975	4. 6468E−02	−0. 475	1. 7793E−01	1. 100	1. 4097E−01
−3. 450	1. 2638E−05	−1. 950	4. 9010E−02	−0. 450	1. 7846E−01	1. 200	1. 3673E−01
−3. 425	1. 6616E−05	−1. 925	5. 1608E−02	−0 425	1. 7893E−01	1. 300	1. 3252E−01
−3. 400	2. 1694E−05	−1. 900	5. 4259E−02	−0. 400	1. 7934E−01	1. 400	1. 2834E−01
−3. 375	2. 8128E−05	−1. 875	5. 6958E−02	−0. 375	1. 7970E−01	1. 500	1. 2423E−01
−3. 350	3. 6227E−05	−1. 850	5. 9703E−02	−0 350	1. 7999E−01	1. 600	1. 2018E−01
−3. 325	4. 6353E−05	−1. 825	6. 2488E−02	−0. 325	1. 8023E−01	1. 700	1. 1622E−01
−3. 300	5. 8931E−05	−1. 800	6. 5311E−02	−0. 300	1. 8042E−01	1. 800	1. 1235E−01
−3. 275	7. 4458E−05	−1. 775	6. 8167E−02	−0. 275	1. 8055E−01	1. 900	1. 0858E−01
−3. 250	9. 3505E−05	−1. 750	7. 1052E−02	−0. 250	1. 8063E−01	2. 000	1. 0492E−01
−3. 225	1. 1673E−04	−1. 725	7. 3961E−02	−0. 225	1. 8066E−01	2. 100	1. 0136E−01
−3. 200	1. 4489E−04	−1. 700	7. 6891E−02	−0. 200	1. 8064E−01	2. 200	9. 7911E−02
−3. 175	1. 7882E−04	−1. 675	7. 9837E−02	−0. 175	1. 8057E−01	2. 300	9. 4573E−02
−3. 150	2. 1949E−04	−1. 650	8. 2795E−02	−0. 150	1. 8046E−01	2. 400	9. 1346E−02
−3. 125	2. 6797E−04	−1. 625	8. 5760E−02	−0. 125	1. 8030E−01	2. 500	8. 8230E−02
−3. 100	3. 2545E−04	−1. 600	8. 8729E−02	−0. 100	1. 8009E−01	2. 600	8. 5223E−02
−3. 075	3. 9326E−04	−1. 575	9. 1696E−02	−0. 075	1. 7984E−01	2. 700	8. 2324E−02
−3. 050	4. 7282E−04	−1. 550	9. 4658E−02	−0. 050	1. 7956E−01	2. 800	7. 9531E−02
−3. 025	5. 6574E−04	−1. 525	9. 7611E−02	−0. 025	1. 7923E−01	2. 900	7. 6842E−02
−3. 000	6. 7372E−04	−1. 500	1. 0055E−01	0. 000	1. 7886E−01	3. 000	7. 4253E−02
−2. 975	7. 9861E−04	−1. 475	1. 0347E−01	0. 025	1. 7845E−01	3. 100	7. 1763E−02
−2. 950	9. 4241E−04	−1. 450	1. 0637E−01	0. 050	1. 7801E−01	3. 200	6. 9368E−02
−2. 925	1. 1072E−03	−1. 425	1. 0925E−01	0. 075	1. 7754E−01	3. 300	6. 7066E−02
−2. 900	1. 2953E−03	−1. 400	1. 1210E−01	0. 100	1. 7703E−01	3. 400	6. 4853E−02
−2. 875	1. 5090E−03	−1. 375	1. 1491E−01	0. 125	1. 7648E−01	3. 500	6. 2725E−02
−2. 850	1. 7509E−03	−1. 350	1. 1769E−01	0. 150	1. 7591E−01	3. 600	6. 0681E−02
−2. 825	2. 0234E−03	−1. 325	1. 2043E−01	0. 175	1. 7530E−01	3. 700	5. 8717E−02
−2. 800	2. 3293E−03	−1. 300	1. 2313E−01	0. 200	1. 7467E−01	3. 800	5. 6830E−02
−2. 775	2. 6714E−03	−1. 275	1. 2579E−01	0. 225	1. 7401E−01	3. 900	5. 5016E−02
−2. 750	3. 0524E−03	−1. 250	1. 2840E−01	0. 250	1. 7332E−01	4. 000	5. 3274E−02
−2. 725	3. 4752E−03	−1. 225	1. 3096E−01	0. 275	1. 7261E−01	4. 100	5. 1599E−02
−2. 700	3. 9428E−03	−1. 200	1. 3346E−01	0. 300	1. 7188E−01	4. 200	4. 9990E−02
−2. 675	4. 4581E−03	−1. 175	1. 3591E−01	0. 325	1. 7112E−01	4. 300	4. 8444E−02
−2. 650	5. 0239E−03	−1. 150	1. 3831E−01	0. 350	1. 7033E−01	4. 400	4. 6959E−02
−2. 625	5. 6431E−03	−1. 125	1. 4065E−01	0. 375	1. 6953E−01	4. 500	4. 5530E−02
−2. 600	6. 3186E−03	−1. 100	1. 4292E−01	0. 400	1. 6871E−01	4. 600	4. 4156E−02
−2. 575	7. 0532E−03	−1. 075	1. 4514E−01	0. 425	1. 6787E−01	4. 700	4. 2836E−02
−2. 550	7. 8494E−03	−1. 050	1. 4729E−01	0. 450	1. 6701E−01		
−2. 525	8. 7098E−03	−1. 025	1. 4937E−01	0. 475	1. 6613E−01		

Table of the Function $\phi(\lambda)$—Continued

λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$
4. 800	4. 1566E–02	10. 800	1. 0337E–02	16. 800	4. 2927E–03	25. 200	1. 8640E–03
4. 900	4. 0344E–02	10. 900	1. 0155E–02	16. 900	4. 2412E–03	25. 400	1. 8337E–03
5. 000	3. 9168E–02	11. 000	9. 9777E–03	17. 000	4. 1905E–03	25. 600	1. 8042E–03
5. 100	3. 8037E–02	11. 100	9. 8044E–03	17. 100	4. 1407E–03	25. 800	1. 7753E–03
5. 200	3. 6948E–02	11. 200	9. 6354E–03	17. 200	4. 0918E–03	26. 000	1. 7471E–03
5. 300	3. 5900E–02	11. 300	9. 4706E–03	17. 300	4. 0437E–03	26. 200	1. 7196E–03
5. 400	3. 4891E–02	11. 400	9. 3097E–03	17. 400	3. 9965E–03	26. 400	1. 6927E–03
5. 500	3. 3919E–02	11. 500	9. 1527E–03	17. 500	3. 9500E–03	26. 600	1. 6664E–03
5. 600	3. 2983E–02	11. 600	8. 9995E–03	17. 600	3. 9043E–03	26. 800	1. 6407E–03
5. 700	3. 2080E–02	11. 700	8. 8499E–03	17. 700	3. 8594E–03	27. 000	1. 6156E–03
5. 800	3. 1210E–02	11. 800	8. 7038E–03	17. 800	3. 8152E–03	27. 200	1. 5910E–03
5. 900	3. 0372E–02	11. 900	8. 5612E–03	17. 900	3. 7718E–03	27. 400	1. 5670E–03
6. 000	2. 9563E–02	12. 000	8. 4219E–03	18. 000	3. 7291E–03	27. 600	1. 5436E–03
6. 100	2. 8783E–02	12. 100	8. 2858E–03	18. 100	3. 6870E–03	27. 800	1. 5206E–03
6. 200	2. 8031E–02	12. 200	8. 1528E–03	18. 200	3. 6457E–03	28. 000	1. 4982E–03
6. 300	2. 7305E–02	12. 300	8. 0229E–03	18. 300	3. 6050E–03	28. 200	1. 4762E–03
6. 400	2. 6604E–02	12. 400	7. 8959E–03	18. 400	3. 5650E–03	28. 400	1. 4547E–03
6. 500	2. 5927E–02	12. 500	7. 7718E–03	18. 500	3. 5257E–03	28. 600	1. 4337E–03
6. 600	2. 5273E–02	12. 600	7. 6505E–03	18. 600	3. 4870E–03	28. 800	1. 4131E–03
6. 700	2. 4642E–02	12. 700	7. 5318E–03	18. 700	3. 4489E–03	29. 000	1. 3929E–03
6. 800	2. 4031E–02	12. 800	7. 4158E–03	18. 800	3. 4114E–03	29. 200	1. 3732E–03
6. 900	2. 3442E–02	12. 900	7. 3024E–03	18. 900	3. 3745E–03	29. 400	1. 3538E–03
7. 000	2. 2872E–02	13. 000	7. 1914E–03	19. 000	3. 3381E–03	29. 600	1. 3349E–03
7. 100	2. 2320E–02	13. 100	7. 0828E–03	19. 100	3. 3024E–03	29. 800	1. 3164E–03
7. 200	2. 1787E–02	13. 200	6. 9766E–03	19. 200	3. 2672E–03	30. 000	1. 2982E–03
7. 300	2. 1271E–02	13. 300	6. 8727E–03	19. 300	3. 2325E–03	30. 200	1. 2804E–03
7. 400	2. 0772E–02	13. 400	6. 7710E–03	19. 400	3. 1984E–03	30. 400	1. 2630E–03
7. 500	2. 0289E–02	13. 500	6. 6714E–03	19. 500	3. 1648E–03	30. 600	1. 2459E–03
7. 600	1. 9821E–02	13. 600	6. 5739E–03	19. 600	3. 1317E–03	30. 800	1. 2292E–03
7. 700	1. 9368E–02	13. 700	6. 4785E–03	19. 700	3. 0991E–03	31. 000	1. 2127E–03
7. 800	1. 8930E–02	13. 800	6. 3850E–03	19. 800	3. 0671E–03	31. 200	1. 1966E–03
7. 900	1. 8504E–02	13. 900	6. 2935E–03	19. 900	3. 0355E–03	31. 400	1. 1809E–03
8. 000	1. 8092E–02	14. 000	6. 2039E–03	20. 000	3. 0043E–03	31. 600	1. 1654E–03
8. 100	1. 7693E–02	14. 100	6. 1161E–03			31. 800	1. 1502E–03
8. 200	1. 7306E–02	14. 200	6. 0301E–03	20. 200	2. 9435E–03	32. 000	1. 1353E–03
8. 300	1. 6930E–02	14. 300	5. 9458E–03	20. 400	2. 8844E–03	32. 200	1. 1207E–03
8. 400	1. 6566E–02	14. 400	5. 8632E–03	20. 600	2. 8270E–03	32. 400	1. 1064E–03
8. 500	1. 6213E–02	14. 500	5. 7823E–03	20. 800	2. 7713E–03	32. 600	1. 0923E–03
8. 600	1. 5870E–02	14. 600	5. 7029E–03	21. 000	2. 7172E–03	32. 800	1. 0786E–03
8. 700	1. 5537E–02	14. 700	5. 6252E–03			33. 000	1. 0650E–03
8. 800	1. 5214E–02	14. 800	5. 5489E–03	21. 200	2. 6646E–03	33. 200	1. 0517E–03
8. 900	1. 4900E–02	14. 900	5. 4742E–03	21. 400	2. 6135E–03	33. 400	1. 0387E–03
9. 000	1. 4595E–02	15. 000	5. 4009E–03	21. 600	2. 5639E–03	33. 600	1. 0259E–03
9. 100	1. 4299E–02	15. 100	5. 3290E–03	21. 800	2. 5156E–03	33. 800	1. 0133E–03
9. 200	1. 4011E–02	15. 200	5. 2584E–03	22. 000	2. 4686E–03	34. 000	1. 0010E–03
9. 300	1. 3731E–02	15. 300	5. 1893E–03	22. 200	2. 4229E–03	34. 200	9. 8884E–04
9. 400	1. 3459E–02	15. 400	5. 1214E–03	22. 400	2. 3784E–03	34. 400	9. 7693E–04
9. 500	1. 3195E–02	15. 500	5. 0548E–03	22. 600	2. 3352E–03	34. 600	9. 6523E–04
9. 600	1. 2938E–02	15. 600	4. 9895E–03	22. 800	2. 2930E–03	34. 800	9. 5374E–04
9. 700	1. 2688E–02	15. 700	4. 9254E–03	23. 000	2. 2520E–03	35. 000	9. 4245E–04
9. 800	1. 2445E–02	15. 800	4. 8625E–03	23. 200	2. 2120E–03	35. 200	9. 3136E–04
9. 900	1. 2208E–02	15. 900	4. 8007E–03	23. 400	2. 1731E–03	35. 400	9. 2046E–04
10. 000	1. 1977E–02	16. 000	4. 7401E–03	23. 600	2. 1352E–03	35. 600	9. 0974E–04
10. 100	1. 1753E–02	16. 100	4. 6805E–03	23. 800	2. 0982E–03	35. 800	8. 9922E–04
10. 200	1. 1535E–02	16. 200	4. 6221E–03	24. 000	2 0622E–03	36. 000	8. 8887E–04
10. 300	1. 1322E–02	16. 300	4. 5647E–03	24. 200	2. 0271E–03	36. 200	8. 7869E–04
10. 400	1. 1115E–02	16. 400	4. 5084E–03	24. 400	1. 9928E–03	36. 400	8. 6869E–04
10. 500	1. 0913E–02	16. 500	4. 4530E–03	24. 600	1. 9594E–03	36. 600	8. 5886E–04
10. 600	1. 0716E–02	16. 600	4. 3986E–03	24. 800	1. 9269E–03	36. 800	8. 4919E–04
10. 700	1. 0524E–02	16. 700	4. 3452E–03	25. 000	1. 8951E–03	37. 000	8. 3968E–04

Table of the Function $\phi(\lambda)$—Continued

λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$	λ	$\phi(\lambda)$
37. 200	8. 3033E–04	45. 600	5. 4406E–04	59. 000	3. 1903E–04	80. 000	1. 7017E–04
37. 400	8. 2113E–04	45. 800	5. 3914E–04	59. 500	3. 1351E–04	80. 500	1. 6800E–04
37. 600	8. 1208E–04	46. 000	5. 3429E–04	60. 000	3. 0813E–04	81. 000	1. 6588E–04
37. 800	8. 0318E–04	46. 200	5. 2951E–04	60. 500	3. 0289E–04	81. 500	1. 6379E–04
38. 000	7. 9442E–04	46. 400	5. 2478E–04	61. 000	2. 9778E–04	82. 000	1. 6174E–04
38. 200	7. 8581E–04	46. 600	5. 2012E–04	61. 500	2. 9280E–04	82. 500	1. 5973E–04
38. 400	7. 7733E–04	46. 800	5. 1552E–04	62. 000	2. 8794E–04	83. 000	1. 5775E–04
38. 600	7. 6898E–04	47. 000	5. 1098E–04	62. 500	2. 8320E–04	83. 500	1. 5581E–04
38. 800	7. 6077E–04	47. 200	5. 0650E–04	63. 000	2. 7857E–04	84. 000	1. 5391E–04
39. 000	7. 5269E–04	47. 400	5. 0208E–04	63. 500	2. 7406E–04	84. 500	1. 5204E–04
39. 200	7. 4474E–04	47. 600	4. 9771E–04	64. 000	2. 6965E–04	85. 000	1. 5021E–04
39. 400	7. 3691E–04	47. 800	4. 9340E–04	64. 500	2. 6535E–04	85. 500	1. 4841E–04
39. 600	7. 2920E–04	48. 000	4. 8915E–04	65. 000	2. 6115E–04	86. 000	1. 4664E–04
39. 800	7. 2161E–04	48. 200	4. 8495E–04	65. 500	2. 5705E–04	86. 500	1. 4490E–04
40. 000	7. 1413E–04	48. 400	4. 8080E–04	66. 000	2. 5304E–04	87. 000	1. 4319E–04
40. 200	7. 0677E–04	48. 600	4. 7671E–04	66. 500	2. 4913E–04	87. 500	1. 4151E–04
40. 400	6. 9953E–04	48. 800	4. 7267E–04	67. 000	2. 4531E–04	88. 000	1. 3987E–04
40. 600	6. 9239E–04	49. 000	4. 6868E–04	67. 500	2. 4157E–04	88. 500	1. 3825E–04
40. 800	6. 8536E–04	49. 200	4. 6474E–04	68. 000	2. 3792E–04	89. 000	1. 3665E–04
41. 000	6. 7844E–04	49. 400	4. 6084E–04	68. 500	2. 3435E–04	89. 500	1. 3509E–04
41. 200	6. 7161E–04	49. 600	4. 5700E–04	69. 000	2. 3085E–04	90. 000	1. 3355E–04
41. 400	6. 6489E–04	49. 800	4. 5320E–04	69. 500	2. 2744E–04	90. 500	1. 3204E–04
41. 600	6. 5827E–04	50. 000	4. 4945E–04	70. 000	2. 2410E–04	91. 000	1. 3055E–04
41. 800	6. 5175E–04	50. 500	4. 4028E–04	70. 500	2. 2083E–04	91. 500	1. 2909E–04
42. 000	6. 4532E–04	51. 000	4. 3138E–04	71. 000	2. 1764E–04	92. 000	1. 2765E–04
42. 200	6. 3899E–04	51. 500	4. 2275E–04	71. 500	2. 1451E–04	92. 500	1. 2624E–04
42. 400	6. 3275E–04	52. 000	4. 1437E–04	72. 000	2. 1145E–04	93. 000	1. 2485E–04
42. 600	6. 2660E–04	52. 500	4. 0623E–04	72. 500	2. 0845E–04	93. 500	1. 2348E–04
42. 800	6. 2053E–04	53. 000	3. 9833E–04	73. 000	2. 0552E–04	94. 000	1. 2213E–04
43. 000	6. 1456E–04	53. 500	3. 9066E–04	73. 500	2. 0264E–04	94. 500	1. 2081E–04
43. 200	6. 0867E–04	54. 000	3. 8321E–04	74. 000	1. 9983E–04	95. 000	1. 1951E–04
43. 400	6. 0286E–04	54. 500	3. 7596E–04	74. 500	1. 9708E–04	95. 500	1. 1823E–04
43. 600	5. 9713E–04	55. 000	3. 6892E–04	75. 000	1. 9438E–04	96. 000	1. 1696E–04
43. 800	5. 9149E–04	55. 500	3. 6207E–04	75. 500	1. 9173E–04	96. 500	1. 1572E–04
44. 000	5. 8592E–04	56. 000	3. 5541E–04	76. 000	1. 8914E–04	97. 000	1. 1450E–04
44. 200	5. 8043E–04	56. 500	3. 4893E–04	76. 500	1. 8660E–04	97. 500	1. 1330E–04
44. 400	5. 7502E–04	57. 000	3. 4263E–04	77. 000	1. 8411E–04	98. 000	1. 1212E–04
44. 600	5. 6968E–04	57. 500	3. 3649E–04	77. 500	1. 8168E–04	98. 500	1. 1095E–04
44. 800	5. 6441E–04	58. 000	3. 3052E–04	78. 000	1. 7928E–04	99. 000	1. 0980E–04
45. 000	5. 5922E–04	58. 500	3. 2470E–04	78. 500	1. 7694E–04	99. 500	1. 0867E–04
45. 200	5. 5410E–04	59. 000	3. 1903E–04	79. 000	1. 7464E–04	100. 000	1. 0756E–04
45. 400	5. 4905E–04	59. 500	3. 1351E–04	79. 500	1. 7239E–04		

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6. References

[1] L. Landau, On the energy loss of fast particles by ionization, J. Physics **8** (1944), 201–205.
[2] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, Higher transcendental functions, Vol. I, New York-Toronto-London, 1953.
[3] F. B. Hildebrand, Introduction to numerical analysis, New York-Toronto-London, 1956.

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Analyticity and Probability Properties of One-Dimensional Brownian Motion

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The most general solutions of one-dimensional Brownian motion, governed by the Chapman-Kolmogoroff functional equation, in the form of series of products of two Laguerre polynomials are described. Some analytical and probabilistic properties of these solutions are discussed and it is shown that these solutions are different from those given by A. N. Kolmogoroff. The limiting behavior of the probability function is investigated. A derivation of the corresponding parabolic partial differential equation is obtained.

1. Introduction

The mathematical study of Brownian motion and diffusion phenomena in the case of a nonuniform fluid led S. Chapman [1] [1] to introduce the functional equation

$$f(\rho,\tau+\tau_0,r,t)=\int f(\rho',\tau,r,t)f(\rho-\rho',\tau_0,r+\rho',t+\tau)d\rho'. \tag{1}$$

B. Hostinsky [11] in studying the functional equation of M. Smoluchowski [19] for the same problem, observed that the functional equation of Chapman was a generalization of the functional equation of Smoluchowski, and found the general solution of the Chapman equation by using the method of integration by substitutions due to V. Volterra.

The same author [11] puts the functional equation of Chapman in the more symmetric form

$$f(x_1,x_3,t_1,t_3)=\int_a^b f(x_1,x_2,t_1,t_2)f(x_2,x_3,t_2,t_3)dx_2 \tag{2}$$

which is equivalent to that of Chapman for the case where the space in which the Brownian motion or diffusion phenomena take place has only one dimension.

On the other hand, A. N. Kolmogoroff [17] has considered directly a very general functional equation valid for a domain of any number of dimensions, which reduces to eq (2) in the case of a single dimension and which can be considered as a very general form of one of the conditions which occurs in the problem of chain probabilities studied by Markoff.

Moreover, A. Kolmogoroff has given particular solutions of eq (2) and has shown that very wide classes of his solutions satisfy linear partial differential equations of the second order and parabolic type, and generalized the functional equation of Chapman from the special problems of Brownian motion and diffusion phenomena to chain probabilities. The eq (2) is commonly called the Chapman-Kolmogoroff functional equation.

Nevertheless, the most general solution of the Chapman-Kolmogoroff equation has not been given. M. Fréchet, however, has given very general solutions having forms different from the forms given by the preceding authors.

M. Fréchet has found two methods: the first [6] is based on a procedure used in mathematical physics and the second [7] on reducing the study of the most general solutions which are doubly square integrable to the corresponding study in the discontinuous case. The two methods provide two distinct forms whose equivalence is neither obvious nor certain.

The most general solutions of the Chapman-Kolmogoroff equation in one-dimensional space have been given by M. Fréchet and the present author. In fact, the case of the finite interval $(0,2\pi)$ has been treated by M. Fréchet [6], and the case of a region infinite in both directions has been investigated in detail [8, 10].

[1] Figures in brackets indicate the literature references at the end of this paper.

In this paper some analytical and probabilistic properties of the most general solutions of one-dimensional Brownian motion in the interval $(0,\infty)$ are described. The corresponding parabolic partial differential equation, which constitutes the fundamental equation of Brownian motion and diffusion theory, is obtained, and the limiting behavior of the solutions at infinity is analyzed.

2. Statement of the Problem and Formulation of the Fundamental Results

Let us consider a free spherical Brownian particle (i.e., in the absence of an external field force) depending on a finite number ν of parameters and which can assume all states Q, forming a certain set V (which one can consider as a region of ν-dimensional Euclidean space). One assumes then that there exists a probability $\omega(M,s;\ \nu,t)$ that the Brownian particle passes from state M at the instant s to any one of the states of a set ν at the later instant t.

The probability function $\omega(M,s;\ \nu,t)$ is, by virtue of the total probability theorem, an additive function of the set ν. It is clear that

$$\omega(M,s;\ \nu,s)=\delta(M,\nu)$$

where

$$\delta=\begin{cases}1 & \text{if } M\epsilon\nu\\ 0 & \text{if } M\not\epsilon\nu.\end{cases}$$

When $s<t$, one can consider the simple case where the additive function of the set ν can be represented by a Lebesgue integral

$$\omega(M,s;\ \nu,t)=\int_\nu f(M,s;\ Q,t)dQ \tag{3}$$

where dQ is an element of the ν-dimensional space and $\int_\nu$ is a ν-tuple integral.

When this assumption is made, one finds, in applying total and compound probability theorems that the Brownian displacement function f, which govern the ν-dimensional Brownian motion,[2] satisfies the Chapman-Kolmogoroff functional equation

$$f(M,s;\ P,t)=\int_V f(M,s;\ Q,u)f(Q,u;\ P,t)dQ \tag{4}$$

for $s<u<t$. V denotes a ν-dimensional Euclidean space and f is Lebesgue-measurable in V.

In the problem of chain probabilities, $f(M,s;\ P,t)dP$ is the elementary probability that the state of a physical system, fixed by ν parameters, and represented by M at the instant s, becomes at the later instant t one of the states of a certain elementary set dP of states. However, in this interpretation the transition probability or density probability function $f(M,s;\ P,t)$ must satisfy the probabilistic conditions

$$f(M,s;\ P,t)\geq 0, \tag{5}$$

$$\int_V f(M,s;\ P,t)dP=1. \tag{6}$$

If the physical system is moving continuously it has to travel the finite distance MP during a finite time t different from s. Then if the states M and P are two different states we have

$$\lim_{t\to s} f(M,s;\ P,t)=0, \qquad M\neq P. \tag{7}$$

If the state P coincides with the state M, the transition probability f is no more continuous and it tends to infinity as $t\to s$

$$\lim_{P\to M,\,t\to s} f(M,s;\ P,t)=\infty. \tag{8}$$

[2] By a ν-dimensional Brownian motion we understand a ν-dimensional strong Markoff process whose projections are all one-dimensional Brownian motion.

Such properties of the transition probability function f, when $t \to s$, resemble those of the Dirac δ-function which is a kind of improper function. It follows that the condition (8) must be excluded from chain probabilities. Therefore the assumption of the existence of an elementary probability $f(M,s;\ P,t)dP$ cannot be accepted completely and an exception must be made for the case $s=t$, $M=P$. If f is interpreted as a generalized function, in the sense of the distribution theory, then the case $s=t$ and $M=P$ is covered by the statement that f has δ-function behavior in this case.

Using the additive theory, both N. Wiener [23] and P. Lévy [18] obtained very important results concerning the transition probability function. M. Kac [12], S. Kakutani [14, 15] and others in a series of papers obtained properties of the Brownian motion in a ν-dimensional space and in connection with the Riemann surfaces. An interesting probabilistic discussion of the general diffusion process in one dimension is also given by W. Feller [5]. Recently M. Kac [13], relying on the properties of Wiener measure in the space of continuous functions, showed the disadvantage of Wiener measure theory applied to Brownian motion.

On the purely physical side the two papers by Uhlenbeck and Ornstein [21] and Uhlenbeck and M. C. Wang [22] should be mentioned.

3. Method of Solution

To obtain a very general solution of the functional eq (4), M. Fréchet [6] tried a method of separation of variables in the form

$$f(M,s;P,t)=\sum_{i=1}^{n} A_i(M,s)B_i(P,t) \tag{9}$$

and found that the necessary and sufficient condition that the expression (9) should be a solution of eq (4) is that, for the values of u such that $s<u<t$, the functions $A_i(Q,u)$ and $B_i(Q,u)$ form a biorthonormal system over V, i.e.,

$$\int_V A_i(Q,u)B_j(Q,u)dQ=\delta_{ij} \text{ where } \delta_{ij}=\begin{cases}1 & \text{if } i=j\\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

Nevertheless, Fréchet realized that the expression (9) does not satisfy the conditions of chain probabilities and introduced instead the infinite series

$$f(M,s;P,t)=\sum_{n=0}^{\infty} A_n(M,s)B_n(P,t) \tag{11}$$

and proved that the necessary and sufficient condition that the transition probability function $f(M,s;P,t)$, satisfying the Chapman-Kolmogoroff eq (4), could be written, in at least one way, in the form (11) where the convergence is uniform for $s<t$, is that the functions $A_i(M,s)$, $B_i(P,t)$ form a complete orthonormal set over V.

The series solution (11) is supposed to be uniformly convergent for s and t fixed such that $s<t$, and M,P varying arbitrarily over V.

4. One-Dimensional Case

Considering the case of one-dimensional space ($\nu=1$), and with the usual notations x, y, and z for M, P, and Q, the Chapman-Kolmogoroff eq (4) can be written in the form

$$f(x,s;y,t)=\int_V f(x,s;z,u)f(z,u;y,t)dz, \qquad s<u<t. \tag{12}$$

Fréchet's method led us to take for the solution of eq (12) the infinite series

$$f(x,s;y,t)=\sum_{n=0}^{\infty} A_n(x,s)B_n(y,t)$$

where the functions A_n, B_n form a biorthonormal set over V.

The case of the finite interval $V(0,2\pi)$ has been treated by M. Fréchet [6]. The solution of eq (12) on $(-\infty,\infty)$ in the form of a series of products of two Hermite polynomials has been given by the author [8, 10], and its analytical and probabilistic properties are discussed in detail.

The most general solution of eq (12) in the interval $(0,\infty)$ is also given by the author [8, 9] in the form

$$f(x,s;y,t)=\sum_{n=0}^{\infty}\psi_n^{(\alpha)}(x)\psi_n^{(\alpha)}(y)\theta^n(s,t)=\exp\left(-\frac{x+y}{2}\right)(xy)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\lambda_n^{-1}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)\theta^n(s,t),\ \alpha>-1, \tag{13}$$

where the infinite sequence of L^2-functions $\left\{\psi_n^{(\alpha)}(x)\right\}$

$$\psi_n^{(\alpha)}(x)=\lambda_n^{-\frac{1}{2}}\exp\left[-\frac{x}{2}\right]x^{\frac{\alpha}{2}}L_n^{(\alpha)}(x)$$

form a complete orthonormal set over $(0,\infty)$, and $L_n^{(\alpha)}(x)$ is the nth associated Laguerre polynomial defined by

$$L_n^{(\alpha)}(x)=\frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}(x^{n+\alpha}e^{-x}).$$

In addition,

$$\theta(s,t)=\frac{a(s)}{a(t)}<1, \qquad \text{for } s<t,$$

$a(s)$ is a positive increasing continuous function $\neq 0$, chosen arbitrarily, and

$$\lambda_n=\frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)},$$

where Γ denotes the gamma function.

It has been shown [8, 9] that, for s,t fixed such that $s<t$ and x,y varying arbitrarily over $(0,\infty)$, the series solution (13) is absolutely convergent in $(0,\infty)$ and uniformly convergent in (ϵ,∞), ϵ being an arbitrarily small but fixed positive number.

Using Hardy's formula

$$\sum_{n=0}^{\infty}\lambda_n^{-1}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)\theta^n=(1-\theta)^{-1}(\theta xy)^{-\frac{\alpha}{2}}\exp\left[-\frac{\theta(x+y)}{1-\theta}\right]I_\alpha\left[\frac{2(\theta xy)^{1/2}}{1-\theta}\right],$$

where I_α denotes the modified Bessel function of the first kind, as the generating function for the series solution (13), we get

$$f(x,s;y,t)=\theta^{-\frac{\alpha}{2}}(1-\theta)^{-1}\exp\left[-\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]I_\alpha\left[\frac{2(\theta xy)^{1/2}}{1-\theta}\right]. \tag{14}$$

The modified Bessel function $I_\alpha(z)$ is defined by

$$I_\alpha(z)=i^{-\alpha}J_\alpha(iz)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}z)^{\alpha+2n}}{n!(\alpha+n)!}$$

where z is positive and α is an integer.

For α a positive integer or zero, a condition which is not incompatible with the assumption $\alpha>-1$, $I_\alpha(z)$ is positive. Thus, according to (14), the transition probability f is always positive for any values of x and y in the interval $(0,\infty)$. Therefore, the solution (13) satisfies the condition (5) provided α takes positive integral or zero values only.

It has been shown [8, 9] that, as a consequence of Kogbetliantz's inequalities [16], the solution

(13) satisfies the special probabilistic condition

$$\lim_{t\to s} f(x,s;\,y,t)=0,$$

for any finite and positive values of x and y, and also the condition

$$\lim_{y\to x,\ t\to s} f(x,s;\,y,t)=\infty.$$

5. Limiting Behavior of the Probability

Let us investigate the asymptotic behavior of the probability $\omega(x,s;\,V,t)$, defined by (3), when the time t tends to infinity. To do so the existence of the limit of the transition probability f defined by the uniformly convergent series

$$f(x,s;\,y,t)=\exp\left[-\frac{x+y}{2}\right](xy)^{\frac{\alpha}{2}}\left\{\Gamma^{-1}(\alpha+1)+\sum_{n=1}^{\infty}\lambda_n^{-1}\left[\frac{a(s)}{a(t)}\right]^n L_n^{(\alpha)}(x)\,L_n^{(\alpha)}(y)\right\} \tag{13'}$$

should be sought. Having in mind that in the factor $\theta(s,t)=a(s)/a(t)$ the function $a(t)$ is a positive increasing continuous function of t, two cases may occur:

1. $$\lim_{t\to\infty} a(t)=\infty$$

hence

$$\lim_{t\to\infty}\theta(s,t)=0.$$

Therefore, for x, y, and s fixed and t tending to infinity we have

$$\lim_{t\to\infty} f(x,s;\,y,t)=\Gamma^{-1}(\alpha+1)(xy)^{\frac{\alpha}{2}}\exp\left[-(x+y)/2\right]$$

which depends on x and y. It follows from (3) that the limit of the probability $\omega(x,s;\,V,t)$, where V is the interval $(0,\infty)$, is given by

$$\lim_{t\to\infty}\omega(x,s;\,V,t)=2^{\frac{\alpha}{2}+1}\,\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma^{-1}(\alpha+1)x^{\frac{\alpha}{2}}\exp\left[-\frac{x}{2}\right]$$

provided $\alpha>-2$. This limit depends on the initial state x which is called a nonoscillatory case. This terminology is due to M. Fréchet, and it means that the probability ω tends to a limit, in the ordinary sense, when t tends to infinity.

By assumption, α takes positive integers or zero values, and therefore the condition $\alpha>-2$ is fulfilled.

2. The only alternative to the case 1 is that

$$\lim_{t\to\infty} a(t)=d$$

where d is finite and positive. In this case

$$\lim_{t\to\infty}\theta(s,t)=a(s)/d=a_1(s)$$

where $0<a_1(s)<1$.

It follows that

$$\lim_{t\to\infty} f(x,s;\,y,t)=(xy)^{\frac{\alpha}{2}}\exp\left[-\frac{x+y}{2}\right]\sum_{n=0}^{\infty}\lambda_n^{-1}[a_1(s)]^n L_n^{(\alpha)}(x)L_n^{(\alpha)}(y),$$

hence

$$\lim_{t\to\infty}\ \omega(x,s;V,t)=\int_V (xy)^{\frac{\alpha}{2}}\exp\left[-\frac{x+y}{2}\right]\sum_{n=0}^{\infty}\lambda_n^{-1}[a_1(s)]^n L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)dy$$

$$=\sum_{n=0}^{\infty}K_n[a_1(s)]^n\psi_n^{(\alpha)}(x)$$

where

$$K_n=\int_0^{+\infty}\psi_n^{(\alpha)}(y)dy.$$

Therefore the limit of the probability ω depends completely on the initial state x at the instant s. This second case is also a nonoscillatory case.

The above results can be stated as follows:

THEOREM. *If the transition probability $f(x,s;y,t)$ satisfying the Chapman-Kolmogoroff eq (12), is in the form (13), then the limit of the probability ω, when t tends to infinity, exists and depends on the initial state, i.e., the probability is nonoscillatory, provided α takes zero or positive integral values.*

6. Stochastic Partial Differential Equations

A. Einstein [3] showed that the probability density f satisfies a partial differential equation of diffusion type whose coefficient is a certain diffusion constant related to the viscosity and temperature of the medium in which Brownian motion takes place, and to the radius of the Brownian particle (assumed spherical).

The existence of the probability density f has been established by many authors from different viewpoints, especially by A. N. Kolmogoroff [17] and W. Feller [4]. Kolmogoroff proved that any function which satisfies the Chapman-Kolmogoroff equation satisfies also a second-order partial differential equation of parabolic type, and he obtained a class of particular solutions and showed that these, subject to certain special conditions concerning the order of the magnitude of moments, satisfy, in variables x and s, the equation

$$B^2(x,s)\frac{\partial^2 f}{\partial x^2}+A(x,s)\frac{\partial f}{\partial x}+\frac{\partial f}{\partial s}=0 \tag{15}$$

where

$$B^2(x,t)=\lim_{\Delta t\to 0}\frac{1}{2\Delta t}\int_{-\infty}^{+\infty}(y-x)^2 f(x,t,t+\Delta t,y)dy,$$

$$A(x,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_{-\infty}^{+\infty}(y-x)f(x,t,t+\Delta t,y)dy,$$

under the assumption that the ratio of the moment of order 3 to that of order 2 tends to zero at the limit $\Delta t=0$.

Kolmogoroff's partial differential equation was derived by C. M. Tchen [20] independently of Kolmogoroff's method.

It can be shown that the solution (13) is different from those discussed by Kolmogoroff.

To see this, note that if the solution (13) belongs to those of Kolmogoroff, then there would exist two functions A and B such that the solution (13) satisfies a partial differential equation of type (15). Calculating the derivatives of f, given by (13), we get

$$\frac{\partial f}{\partial s}=(xy)^{\frac{\alpha}{2}}\exp\left[-\frac{x+y}{2}\right]\sum_{n\geq 0}n\frac{\theta^{n-1}}{\lambda_n}\frac{\partial\theta}{\partial s}L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)$$

$$\frac{\partial f}{\partial x}=(xy)^{\frac{\alpha}{2}}\exp\left[-\frac{x+y}{2}\right]\sum_{n\geq 0}\frac{\theta^n}{\lambda_n}\left[\left(\frac{\alpha}{2x}-\frac{1}{2}\right)L_n^{(\alpha)}(x)+L_n'^{(\alpha)}(x)\right]L_n^{(\alpha)}(y)$$

$$\frac{\partial^2 f}{\partial x^2}=(xy)^{\frac{\alpha}{2}}\exp\left[-\frac{x+y}{2}\right]\sum_{n\geq 0}\frac{\theta^n}{\lambda_n}\left[L_n''^{(\alpha)}(x)+\left(\frac{\alpha}{x}-1\right)L_n'^{(\alpha)}(x)+\left(\frac{x^2}{4}+\frac{\alpha^2-2\alpha}{4x^2}-\frac{\alpha+1}{2}\right)L_n^{(\alpha)}(x)\right]L_n^{(\alpha)}(y),$$

where prime denotes the derivative with respect to x.

For x and y varying in $(0,\infty)$, t fixed, and s lying in a bounded interval such that $s<t$, the series, obtained by differentiating (13) term by term, are uniformly convergent. Substituting the above derivatives in (15), one gets

$$\exp\left[-\frac{x+y}{2}\right](xy)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\frac{\theta^n}{\lambda_n}\left[U_n(x,s,t)\right]L_n^{(\alpha)}(y)=0 \tag{16}$$

where $U_n(x,s,t)$ denotes

$$U_n(x,s,t)=\frac{n}{\theta}\frac{\partial\theta}{\partial s}L_n^{(\alpha)}(x)+A(x,s)\left[L_n'^{(\alpha)}(x)+\left(\frac{\alpha}{2x}-\frac{x}{2}\right)L_n^{(\alpha)}(x)\right]$$
$$+B^2(x,s)\left[L_n''^{(\alpha)}(x)+\left(\frac{\alpha}{x}-x\right)L_n'^{(\alpha)}(x)+\left(\frac{x^2}{4}+\frac{\alpha^2-2\alpha}{4x^2}-\frac{\alpha+1}{2}\right)L_n^{(\alpha)}(x)\right].$$

Since the relation (16) must hold for any value of y in $(0,\infty)$, the expression in the square brackets must be zero for any positive integer value of n; that is, after dividing by $L_n^{(\alpha)}(x)$,

$$\frac{n}{\theta}\frac{\partial\theta}{\partial s}+A\left[\frac{L_n'^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}+\left(\frac{\alpha}{2x}-\frac{x}{2}\right)\right]+B^2\left[\frac{L_n''^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}+\left(\frac{\alpha}{x}-x\right)\frac{L_n'^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}+\left(\frac{x^2}{4}+\frac{\alpha^2-2\alpha}{4x^2}-\frac{\alpha+1}{2}\right)\right]=0. \tag{17}$$

As $\frac{1}{\theta}\frac{\partial\theta}{\partial s}$ is independent of n and x, it is clear that the relation (17) does not hold unless the two coefficients $A(x,s)$ and $B^2(x,s)$ are both zero. Therefore $\theta(s,t)=a(s)/a(t)$ will be independent of s which is contrary to the assumption that $a(s)$ is a positive increasing continuous function of s. This proves the statement.

It can also be shown by computation that the relation (17) is possible if the two functions $A(x,s)$ and $B(x,s)$ are both zero. In fact for $n=0$, we have

$$\frac{A}{B^2}=\frac{x^4+\alpha^2-2\alpha-2x^2(\alpha+1)}{2x(x^2-\alpha)}. \tag{18}$$

On the other hand, setting $n=1$ and $n=2$ in relation (17) and eliminating $\frac{1}{\theta}\frac{\partial\theta}{\partial s}$ between the two relations obtained, one gets

$$\frac{A}{B^2}=\frac{C}{D} \tag{19}$$

where

$$C=\frac{2(\alpha-x^2)}{x(x-\alpha-1)}+\frac{x^4+\alpha^2-2\alpha}{4x^2}-\frac{\alpha+1}{2}-\frac{2x+2(\alpha-x^2)(x-\alpha-2)}{x^3-2(\alpha+2)x^2+(\alpha+2)(\alpha+1)x}$$

and

$$D=\frac{2(x-\alpha-2)}{x^2-2(\alpha+2)x+(\alpha+2)(\alpha+1)}-\frac{2}{(x-\alpha-1)}+\frac{x}{2}.$$

The two ratios (18) and (19) are different, hence the relation (17) does not hold unless the two functions A and B are both zero.

Now we propose to obtain a derivation of the corresponding parabolic partial differential equation or the so-called backward equation of diffusion theory. For this purpose we can use the generating function (14) and get

$$I_\alpha(z)=\theta^{\frac{\alpha}{2}}(1-\theta)\exp\left[\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]f(x,s;y,t) \tag{20}$$

where

$$z=2(1-\theta)^{-1}(\theta xy)^{1/2},$$

and $I_\alpha(z)$ satisfies

$$z^2I_\alpha''(z)+zI_\alpha'(z)-(\alpha^2+z^2)I_\alpha(z)=0 \tag{21}$$

prime denotes the derivative with respect to z.

Taking the derivatives of (20) with respect to x and computing the different terms of (21), one gets

$$zI'_\alpha = 2x\theta^{\frac{\alpha}{2}}(1-\theta)\exp\left[\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]\left[\frac{\partial f}{\partial x}+\frac{1+\theta}{2(1-\theta)}f\right],$$

$$z^2I''_\alpha = 4x^2\theta^{\frac{\alpha}{2}}(1-\theta)\exp\left[\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]\left\{\frac{\partial^2 f}{\partial x^2}+\frac{\partial f}{\partial x}\left(\frac{1+\theta}{1-\theta}+\frac{1}{2x}\right)+f\left[\frac{(1+\theta)^2}{4(1-\theta)^2}+\frac{1+\theta}{4x(1-\theta)}\right]\right\},$$

$$(\alpha^2+z^2)I_\alpha = \theta^{\frac{\alpha}{2}}(1-\theta)\exp\left[\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]\frac{\alpha^2(1-\theta)^2+4\theta xy}{(1-\theta)^2}f.$$

Substituting in (21), and dividing by $\theta^{\frac{\alpha}{2}}(1-\theta)\exp\left[\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]$ we have

$$4x^2\frac{\partial^2 f}{\partial x^2}+4x\left[\frac{x(1+\theta)}{1-\theta}+1\right]\frac{\partial f}{\partial x}+\frac{x^2(1+\theta)^2+2x(1-\theta^2)-\alpha^2(1-\theta)^2-4\theta xy}{(1-\theta)^2}f=0. \tag{22}$$

The relations

$$\frac{\partial I_\alpha}{\partial x}=I'_\alpha\frac{\partial z}{\partial x} \text{ and } \frac{\partial I_\alpha}{\partial \theta}=I'_\alpha\frac{\partial z}{\partial \theta}$$

lead us to express f in terms of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial \theta}$ as follows:

$$f=\frac{2(1-\theta)}{(1+\theta)x-\alpha(1-\theta)^2+2\theta(1-\theta)-2\theta(x+y)}\left(\theta\frac{\partial f}{\partial \theta}-x\frac{\partial f}{\partial x}\right).$$

Substituting the above expression of f in (22) and setting

$$\beta(x,s)=2x^2(1-\theta)[(1+\theta)x-\alpha(1-\theta)^2+2\theta(1-\theta)-2\theta(x+y)][x^2(1+\theta)^2 +2x(1-\theta^2)-\alpha^2(1-\theta)^2-4\theta xy]^{-1}$$

and

$$\gamma(x,s)=x\{2[x(1+\theta)+(1-\theta)][(1+\theta)x-\alpha(1-\theta)^2+2\theta(1-\theta)-2\theta(x+y)][x^2(1+\theta)^2 +2x(1-\theta^2)-\alpha^2(1-\theta^2-4\theta xy]^{-1}-1\},$$

one gets

$$\beta(x,s)\frac{\partial^2 f}{\partial x^2}+\gamma(x,s)\frac{\partial f}{\partial x}+\theta\frac{\partial f}{\partial \theta}=0. \tag{23}$$

On the other hand

$$\frac{\partial f}{\partial s}=\frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial s}=\frac{a'(s)}{a(t)}\frac{\partial f}{\partial \theta}$$

so that

$$\theta\frac{\partial f}{\partial \theta}=\frac{a(s)}{a(t)}\frac{a(t)}{a'(s)}\frac{\partial f}{\partial s}=\frac{a(s)}{a'(s)}\frac{\partial f}{\partial s}$$

where prime denotes the derivative with respect to s. Hence equation (23) becomes

$$\beta\frac{\partial^2 f}{\partial x^2}+\gamma\frac{\partial f}{\partial x}+\frac{a(s)}{a'(s)}\frac{\partial f}{\partial s}=0. \tag{24}$$

This derivation presupposes certain differentiability conditions on f and $a(s)$, namely: f must have continuous partial derivatives of order ≤ 2 with respect to x and s, and the right-handed derivative $a'(s)$ must exist. Therefore the following theorem can be stated:

THEOREM. *If the transition probability function f, defined by (13), is of the class C^2, and the right-handed derivative $a'(s)$ of the increasing continuous function $a(s)$ exists, and is everywhere positive and different from zero, then the transition probability f satisfies, in the variables x and s, the parabolic equation (24).*

It can be observed that the parabolic eq (24) to which the solution (13) satisfies is different from equation (15) given by Kolmogoroff.

The coefficient of $\frac{\partial f}{\partial s}$ in (24) suggests setting $a(s)=e^s$, and then putting $a(t)=e^t$, we have

$$\theta=e^{s-t}.$$

The special case $a(s)=e^s$ leads us to a particular solution of eq (12) in the form

$$f(x,s;\,y,t)=\exp\,[-(x+y)/2](xy)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\lambda_n^{-1}\exp\,[n(s-t)]L_n^{(\alpha)}(x)L_n^{(\alpha)}(y)$$

$$=\theta^{-\frac{\alpha}{2}}(1-\theta)^{-1}\exp\left[-\frac{(1+\theta)(x+y)}{2(1-\theta)}\right]I_\alpha\left[\frac{2(\theta xy)^{\frac{1}{2}}}{1-\theta}\right], \tag{25}$$

and the eq (24) will be reduced to the normal form

$$\beta\frac{\partial^2 f}{\partial x^2}+\gamma\frac{\partial f}{\partial x}+\frac{\partial f}{\partial s}=0. \tag{26}$$

If we now set, more generally,

$$a(s)=e^S$$

where the variable S increases with s, then

$$a(t)=e^T$$

and

$$s=\mu(S)$$

$$t=\mu(T)$$

where $\mu(S)$ is an increasing continuous function of S. We then have

$$\theta=e^{S-T}.$$

The function $f(x,s;\,y,t)$ becomes $F(x,S;\,y,T)$, and the Chapman-Kolmogoroff equation will still be satisfied if the instants s, t are replaced by numbers S and T and the function $F(x,S;\,y,T)$ can be written as follows:

$$F(x,S;\,y,T)=\int_0^\infty F(x,S;\,z,U)F(z,U;\,y,T)dz,\qquad S<U<T. \tag{27}$$

The series solution (13) now becomes

$$f(x,s;\,y,t)=f[x,\mu(S);\,y,\mu(T)]=F(x,S;\,y,T)$$

$$=\exp\left[-\frac{x+y}{2}\right](xy)^{\frac{\alpha}{2}}\sum_{n=0}^{\infty}\lambda_n^{-1}\exp\,[n(S-T)]L_n^{(\alpha)}(x)L_n^{(\alpha)}(y). \tag{28}$$

When the time $t\to\infty$, T tends either to infinity or (T being negative as well as S) to zero, then θ tends either to zero or e^S, so that the limiting behavior of the probability function ω, defined by

$$\omega=\int_0^\infty F(x,S;\,y,T)dy,$$

will still be nonoscillatory.

There is nothing arbitrary in $F(x,S;\,y,T)$, defined by (28), which is a particular solution of (27). Therefore the most general solution will be in the form

$$f(x,s;\,y,t)=F[x,\eta(s);\,y,\eta(t)]$$

where $\eta(s)=\ln a(s)$ is an arbitrary function varying in $(-\infty,\infty)$.

7. References

[1] S. Chapman, Proc. Roy. Soc. **A119,** 34–54 (1928).
[2] J. L. Doob, Trans. Am. Math. Soc. **42,** 107–140 (1937).
[3] A. Einstein, Ann. Phys. **17,** 549 (1905).
[4] W. Feller, Math. Ann. **113,** 113–160 (1936).
[5] W. Feller, Trans. Am. Math. Soc. **77,** 1–31 (1954).
[6] M. Fréchet, Proc. London Math. Soc. **39,** 515–540 (1935).
[7] M. Fréchet, Compt. Rend. Acad. Sci. Paris **200,** 369 (1935).
[8] A. Ghaffari, Doctoral Thesis, Math., Paris Univ. (1936), Gauthier-Villars, Paris.
[9] A. Ghaffari, Bull. Am. Math. Soc. **58,** 43 (1952).
[10] A. Ghaffari, Proc. of the Inter. Conf. on Partial Differential Equations and Continuum Mechanics, 348–350, held at the University of Wisconsin, Madison, Wis., June 1960. The Univ. of Wisconsin Press, 1961.
[11] B. Hostinsky, Mémorial des Sciences Math., Fasc. **52** (1931).
[12] M. Kac, Am. Math. Monthly **54,** 369–391 (1947).
[13] M. Kac, Lectures in Applied Math., Proc. of the Summer Seminar, Boulder, Colo. (1957), Vol. **I,** Interscience Publishers.
[14] S. Kakutani, Proc. Japan Acad. **21,** 138 (1945).
[15] S. Kakutani, Proc. Nat. Acad. Sci. U.S.A. **36,** 319 (1950).
[16] E. Kogbetlintz, J. Math. and Phys. **14,** 1–99 (1935).
[17] A. N. Kolmogoroff, Math. Ann. **104,** 415–458 (1931).
[18] P. Lévy, Am. J. Math. Vol. **62,** 487 (1948).
[19] M. Smoluchovski, Phys. Z. **17,** 557–571 (1916).
[20] C. M. Tchen, Vers. Ned. Akad. V. Wetensch., Afd. Natuurk. **53,** 400–410 (1944).
[21] G. E. Uhlenbeck and L. S. Ornstein, Phys. Rev. **36,** 823–841 (1930).
[22] M. C. Wang and G. E. Uhlenbeck, Rev. Modern Phys. **17,** 323–342 (1945).
[23] N. Wiener, Publications of M.I.T., series II, No. 60 (1923).

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Some Higher Order Integral Identities With Application to Bounding Techniques*

J. H. Bramble [1] and B. E. Hubbard [2]

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Let R be a simply connected region in E_N with smooth bounding surface S. For a sufficiently differentiable set of functions we derive a class of quadratic integral identities relating surface integrals of derivatives to integrals over R. These identities are a generalization of a first order identity given by L. Hörmander (Compt. Rend. Douzième Congr. des Mathématiciens Scandinaves Tenu à Lund, 1953, pp. 105–115) and L. E. Payne and H. F. Weinberger (Pacific J. Math. (1958) pp. 551–573). As an example of an application of these identities we consider a solution u of the boundary value problem $\Delta u - pu = F$ in R and $u = f$ on S. Here Δ denotes the Laplace operator and $0 \leq p(x)$. We obtain pointwise a priori bounds for the derivatives of u in R in terms of a quadratic functional of an arbitrary function. Hence the Rayleigh-Ritz procedure can be used to make the error arbitrarily small.

1. Introduction

In a recent paper [6] [3] L. E. Payne and H. F. Weinberger give a method for obtaining bounds for solutions of second order elliptic boundary value problems. In that paper they give a generalization (see eq 2.4) of an integral identity of Rellich [7] which was essential to their method. This generalization, which was first obtained and applied to hyperbolic operators by L. Hörmander [4], displays the highest derivatives in the form of a second order operator $Lu \equiv (a^{ij}u_{,j})_{,i}$. This fact makes the identity useful in treating boundary problems for the corresponding differential equation $Lu = F$.

Use was also made of this identity by Hubbard [5] to obtain bounds for membrane eigenvalues by finite difference methods.

In section 2 we give a further generalization of Hörmander's result, which involves higher derivatives on the boundary and where the highest order terms enter as derivatives of Lu. Section 3 gives an application of these higher order identities in obtaining pointwise bounds for the derivatives of a function u. The particular problem treated assumes a knowledge of $\Delta u - pu$ in a region R and u on the boundary C where Δ is Laplace operator and $p > 0$ in $R + C$. Because of the important physical application of this equation (see e.g., Bergman and Schiffer [1]), this problem was chosen as an example to illustrate a use of the higher order identity. More general equations could be treated with only technical modifications. Other methods for obtaining explicit pointwise bounds for derivatives in such problems have been given by J. B. Diaz [3] and by Payne and Weinberger [6]. In both cases the method given involves differentiation of the fundamental solution in Green's third identity, thus introducing more singular behavior of the bounds as the boundary is approached. In our paper the bounds for derivatives of all orders have the same behavior near the boundary, provided the boundary data is sufficiently differentiable.

Extensions of the fundamental identity (2.4) can be derived for higher order operators by the same technique used here. One such, involving the biharmonic operator Δ^2, is applied in a forthcoming paper of J. H. Bramble and L. E. Payne [2] to obtain explicit pointwise bounds in the biharmonic Dirichlet problem.

As has been previously pointed out in various places (see e.g. [6]) bounds of the type obtained here can be used in conjunction with a Rayleigh-Ritz technique to approximate the unknown solution (or derivative) arbitrarily closely.

2. A Class of Quadratic Integral Identities

Let

$$L(u) = (a^{ij}u_{,j})_{|i}, \tag{2.1}$$

be defined on a region R in E_n with boundary C. The symbol $_{|i}$ indicates covariant differentiation which coincides with $_{,i}$ (partial differentiation with respect to x^i) in Cartesian coordinates. A repeated index indicates summation from 1 to N. The symmetric tensor $a^{ij}(x)$ is assumed to possess piecewise continuous derivatives of order $M+1$. In addition, let the eigenvalues of the matrix a^{ij} be bounded away from zero and infinity in R. Hence there exists a positive constant, a, such that for all real nonzero $(\xi_1, \ldots, \xi_n)$ and all x in R,

$$a^{-1} \sum_{i=1}^{N} \xi_i^2 \leq a^{ij}\xi_i\xi_j \leq a \sum_{i=1}^{N} \xi_i^2. \tag{2.2}$$

Under these conditions L is said to be uniformly elliptic.

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[3] Figures in brackets indicate the literature references at the end of this paper.

Let $f^i(x), A_1^{ij}(x), \ldots, A_m^{ij}(x)$ be symmetric tensors of the second order with piecewise continuous first derivatives in R. L. E. Payne and H. F. Weinberger have developed the following first order quadratic identity [6] and used it to obtain pointwise bounds in certain boundary problems. It is clear that

$$\begin{aligned}(f^m a^{ij} u_{|i} u_{|j})_{|m} &= (f^m a^{ij})_{|m} u_{|i} u_{|j} + 2 f^m a^{ij} u_{|i} u_{|mj} \\ &= (f^m a^{ij})_{|m} u_{|i} u_{|j} + 2 (f^m a^{ij} u_{|i} (u_{|m})_{|j} \\ &\quad - 2 f^m_{|j} a^{ij} u_{|i} u_{|m} - 2 f^m u_{|m} L(u). \end{aligned} \tag{2.3}$$

An application of the divergence theorem yields

$$\begin{aligned}\oint_C \{ f^m a^{ij} - 2 f^j a^{im} \} u_{|i} u_{|j} \nu_m ds &= -2 \int_R f^m u_{|m} L(u) dv \\ &\quad + \int_R \{ (f^m a^{ij})_{|m} - 2 f^j_{|m} a^{im} \} u_{|i} u_{|j} dv. \end{aligned} \tag{2.4}$$

In an analogous manner we can develop a homogeneous quadratic integral identity involving second derivatives of u on the boundary C. Now

$$\begin{aligned}&(f^m A^{ij} a^{kl} u_{|ik} u_{|jl})_{|m} \\ &\quad = (f^m A^{ij} a^{kl})_{|m} u_{|ik} u_{|jl} + 2 f^m A^{ij} a^{kl} u_{|ik} u_{|jml} \\ &\quad = + 2 (f^m A^{ij} a^{kl} u_{|ik} u_{|jm})_{|l} - 2 f^m A^{ij} u_{|jm} (Lu)_{|i} \\ &\quad + \{ (f^m A^{ij} a^{kl})_{|m} - 2 (f^l A^{ij} a^{km})_{|m} \} u_{|ik} u_{|jl} \\ &\qquad + 2 f^m A^{ij} (a^{kl}_{|k} u_{|l})_{|i} u_{|jm}. \end{aligned} \tag{2.5}$$

After applying the divergence theorem we have

$$\begin{aligned}&\oint_C A^{kl} \{ f^m a^{ij} - 2 f^j a^{im} \} u_{|ik} u_{|jl} \nu_m ds \\ &= -2 \int_R f^m A^{ij} u_{|jm} (Lu)_{|i} dv + 2 \int_R f^m A^{ij} (a^{kl}_{|l} u_{|k})_{|i} u_{|jm} dv \\ &\quad + \int_R \{ (f^m A^{ij} a^{kl})_{|m} - 2 (f^l A^{ij} a^{km})_{|m} \} u_{|ik} u_{|jl} dv. \end{aligned} \tag{2.6}$$

The corresponding identity which involves covariant derivatives of order $M+1$ on the boundary is

$$\begin{aligned}&\oint_C A_1^{i_1 j_1} \ldots A_M^{i_M j_M} \{ f^m a^{kl} - 2 f^l a^{km} \} u_{|ki_1 \ldots i_M} u_{|lj_1 \ldots j_M} \nu_m dS \\ &= -2 \int_R f^m A_1^{i_1 j_1} \ldots A_M^{i_M j_M} (Lu)_{|i_1 \ldots i_M} u_{|j_1 \ldots j_M m} dv \\ &\quad + \int_R \{ (f^m A_1^{i_1 j_1} \ldots A_M^{i_M j_M} a^{kl})_{|m} \\ &\quad - 2 (f^l A_1^{i_1 j_1} \ldots A_M^{i_M j_M} a^{km})_{|m} \} u_{|ki_1 \ldots i_M} u_{|lj_1 \ldots j_M} dv \\ &\quad + 2 \int_R f^m A_1^{i_1 j_1} \ldots A_M^{i_M j_M} (a^{kl}_{|k} u_{|l})_{|i_1 \ldots i_M} u_{|j_1 \ldots j_M m} dv. \end{aligned} \tag{2.7}$$

Since the integrands are in each case tensor invariants, we may perform each integration in the most advantageous coordinate system. Since boundary conditions usually are given in terms of normal and tangential derivatives we shall display the derivatives appearing in the boundary integrals in such a form. Assume C to be a surface possessing $M+1$ continuous derivatives as given in the parametric form

$$x^i = G^i(y^1, \ldots, y^{N-1}), \qquad i = 1, \ldots, N. \tag{2.8}$$

The unit normal $X^i(y)$ satisfies the conditions

$$\begin{aligned} G^i_{,\alpha} X^i &= 0, \qquad \alpha = 1, \ldots, N-1, \\ X^i X^i &= 1 \end{aligned} \tag{2.9}$$

and the orientation of X^i is taken inward. In what follows Greek indices will always range over $1, \ldots, N-1$, whereas Latin indices will refer to $1, \ldots, N$.

We introduce geodesic normal coordinates $(y^1, \ldots, y^N)$ in a strip immediately adjacent to the bounding surface. The transformation is given by

$$x^i = G^i(y^1, \ldots, y^{N-1}) + y^N X^i(y^1, \ldots, y^{N-1}). \tag{2.10}$$

The coordinate y^N refers to a distance along the inward normal to C at the point $(y \ldots, y^{N-1})$. The metric tensor has the form

$$\begin{aligned} g_{\alpha\beta} &= (G^k_{,\alpha} + y^N X^k_{,\alpha})(G^k_{,\beta} + y^N X^k_{,\beta}), \\ g_{\alpha N} &= (G^k_{,\alpha} + y^N X^k_{,\alpha}) X^k = \tfrac{1}{2} y^N (X^k X^k)_{,\alpha} = 0, \\ g_{NN} &= X^k X^k = 1 \end{aligned} \tag{2.11}$$

The Christoffel symbols which involve the normal coordinate y^N are

$$\begin{aligned} \left\{ \begin{matrix} \rho \\ N\sigma \end{matrix} \right\} &= \frac{1}{2} g^{\rho\mu} \frac{\partial g_{\mu\sigma}}{\partial y^N}, \\ \left\{ \begin{matrix} N \\ \rho\sigma \end{matrix} \right\} &= -\frac{1}{2} \frac{\partial g_{\rho\sigma}}{\partial y^N} \\ \left\{ \begin{matrix} N \\ N\rho \end{matrix} \right\} &= \left\{ \begin{matrix} \rho \\ N,N \end{matrix} \right\} = \left\{ \begin{matrix} N \\ NN \end{matrix} \right\} = 0. \end{aligned} \tag{2.12}$$

The various second order covariant derivatives of u then have the form

$$\begin{aligned} \bar{u}_{|\alpha\beta} &= \frac{\partial^2 \bar{u}}{\partial y^\alpha \partial y^\beta} - \left\{ \begin{matrix} k \\ \alpha\beta \end{matrix} \right\} \bar{u}_{|k}, \\ \bar{u}_{|\alpha N} &= \bar{u}_{|N\alpha} = \frac{\partial^2 \bar{u}}{\partial y^\alpha \partial y^N} - \left\{ \begin{matrix} \beta \\ \alpha N \end{matrix} \right\} \bar{u}_{|\beta}, \\ \bar{u}_{|NN} &= \frac{\partial^2 \bar{u}}{(\partial y^N)^2}. \end{aligned} \tag{2.13}$$

We indicate with a bar when the tensor is expressed in geodesic normal coordinates. Our operator takes the form

$$Lu = (a^{ij} u_{,j})_{|i} = (\bar{a}^{ij} \bar{u}_{,j})_{|i}. \tag{2.14}$$

When L is the Laplace operator Δ we have

$$\Delta u=\delta^{ij}u_{,ij}=\overline{u}_{,NN}+g^{\alpha\beta}\overline{u}_{|\alpha\beta}, \tag{2.15}$$

since $g^{\alpha N}=0$ and $g^{NN}=1$. Also the first invariant takes the form

$$|\text{grad } u|^2=(\overline{u}_{,N})^2+g^{\alpha\beta}\overline{u}_{,\alpha}\overline{u}_{,\beta}. \tag{2.16}$$

We now express the boundary integral appearing in (2.4) in geodesic normal coordinates.

$$-\oint_C\{\overline{f}^N\overline{a}^{ij}-2\overline{f}^i\overline{a}^{iN}\}\overline{u}_{,i}\,\overline{u}_{,j}\,dS=-\oint_C\{\overline{f}^N[(\overline{a}^{\alpha\beta}\overline{u}_{,\alpha}\overline{u}_{,\beta}) -\overline{a}^{NN}(\overline{u}_{,N})^2]-2(\overline{f}^\alpha\overline{u}_{,\alpha})(\overline{a}^{iN}\overline{u}_{,i})\}dS. \tag{2.17}$$

If $L\equiv\Delta$ then $\overline{a}^{ij}=g^{ij}$ and (2.17) becomes

$$-\oint_C\{\overline{f}^N[(g^{\alpha\beta}\overline{u}_{,\alpha}\overline{u}_{,\beta})-(\overline{u}_{,N})^2]-2(\overline{f}^\alpha\overline{u}_{,\alpha})\overline{u}_{,N}\}dS. \tag{2.18}$$

The surface integral in (2.6) can be treated in a similar manner to obtain

$$-\oint_C\overline{A}^{kl}\{\overline{f}^N[\overline{a}^{\alpha\beta}\overline{u}_{|k\alpha}\overline{u}_{|l\beta}-\overline{a}^{NN}(\overline{u}_{|kN})(\overline{u}_{|lN})] -2(\overline{f}^\alpha\overline{u}_{|k\alpha})(\overline{a}^{iN}\overline{u}_{|il})\}dS. \tag{2.19}$$

In particular if $\overline{f}^i,\overline{A}^{kl}$ is chosen on C so that

$$\begin{aligned}f^N&=\overline{A}^{NN}=1,\\ f^\alpha&=\overline{A}^{\alpha j}=0,\end{aligned} \tag{2.20}$$

then for $L\equiv\Delta$ (2.19) takes the simple form

$$-\oint_C\{g^{\alpha\beta}\overline{u}_{|N\alpha}\overline{u}_{|N\beta}-(\overline{u}_{|NN})^2\}dS. \tag{2.21}$$

In view of (2.15) we have thereby isolated an interesting combination of mixed normal and tangential second derivatives on C in terms of second order tangential derivatives on C and certain integrals over R.

In fact, a specific mixed derivative, say $\overline{u}_{|N\sigma}$, can be isolated in this manner. Let

$$\begin{aligned}&\overline{f}^N=1,\quad \overline{f}^\alpha=0,\\ &\overline{A}^{\sigma\sigma}=1,\ \overline{A}^{ij}=0\qquad i,j\neq\sigma\end{aligned} \tag{2.22}$$

Then for $L\equiv\Delta$ (2.19) becomes

$$-\oint_C\{g^{\alpha\beta}\overline{u}_{|\sigma\alpha}\overline{u}_{|\sigma\beta}-(\overline{u}_{|N\sigma})^2\}dS. \tag{2.23}$$

As we shall see in the next section, these choices of $\overline{f}^i,\overline{A}^{kl}$ will enable us to obtain new pointwise bounds for derivatives in certain boundary problems.

The following notation will be used throughout this paper. Let $f(x)$ be a piecewise continuous function on $R+C$, then

$$f_M=\max_{x\in R+C}f(x),\qquad f_m=\min_{x\in R+C}f(x). \tag{2.24}$$

For $N=2$, $x^i=G^i(y^1)$ represents a plane closed curve. Equation (2.10) becomes

$$x^i=G^i(S)+Nn^i(S) \tag{2.25}$$

where we have set $y^1=S$ (arclength along C) and $y^2=N$ (distance along the unit inward normal (n^1, n^2). If $K(S)$ is the curvature of C then we have

$$g^{11}=(1-K(S)N)^{-2},\quad g^{12}=g^{21}=0,\quad g^{22}=1. \tag{2.26}$$

The use of this coordinate system is, of course, restricted to a certain strip in the neighborhood of the boundary where in terms of it all points are uniquely defined.

3. Pointwise Bounds

As was previously noted, Payne and Weinberger [6] made use of (2.4) in order to obtain bounds for solutions of second order elliptic partial differential equations. This identity enabled them to estimate the integral over a closed surface of the square of the normal derivative of a solution of a second order equation in terms of integrals of the squares of the function and its tangential derivative over the surface. In order to bound the derivatives at a point within the region they noted that one could differentiate the "Green's Identity," being careful to define a "parametrix" in such a way that this is permissible. The resulting estimates have the disadvantage that the coefficients become infinite more rapidly as the boundary is approached than do those in the estimates for the value of the solution itself. This procedure has also been used by Diaz [3] in estimating derivatives at a point for solutions of Laplace's equation.

One could proceed in an alternate manner. That is, instead of differentiating the "Green's Identity," simply write the identity using for the function to be evaluated at the point, the derivative of the solution. This leads to the estimation of surface integrals of squares of second derivatives of the function over the surface in terms of surface integrals of squares of tangential derivatives of the function over the surface. The identity (2.4) gives a means of obtaining the necessary inequalities.

As an example of an application of these identities we consider the problem of obtaining a priori pointwise bounds for the derivatives of a function u for which $Lu\equiv\Delta u-pu$ is known in $R+C$ and u is known on C. Here R is a simply connected finite plane region bounded by the smooth closed curve C, Δ is the Laplace operator and $p(x,y)>0$ and

bounded together with its first derivatives in $R+C$. (The function p is taken to be positive in $R+C$ for the sake of convenience. By a slight modification, similar results could be obtained for $p \geq 0$.) We assume that Lu and u are sufficiently smooth in R and on C respectively.

As a starting point we use the "Green's Identity"

$$(P)=\int_R \Gamma \Delta v dA+\oint_C \left(v\frac{\partial \Gamma}{\partial n}-\Gamma\frac{\partial v}{\partial n}\right)dS, \quad (3.1)$$

where v is any sufficiently smooth function in $R+C$, P is a point in the interior of R, $\Gamma=-\frac{1}{2\pi}\ln r_{PQ}$ (r_{PQ} = distance from the point P to another point Q), and $\frac{\partial}{\partial n}$ is the outward normal derivative. Now let $v=u_{,i}\equiv\frac{\partial u}{\partial x_i}$. Then we have

$$u_{,i}(P)=\int_R \Gamma \Delta u_{,i} dA+\oint_C \left(u_{,i}\frac{\partial \Gamma}{\partial n}-\Gamma\frac{\partial u_{,i}}{\partial n}\right)dS, \quad (3.2)$$

which may be written

$$\chi_i(P)\equiv u_{,i}(P)-\int_R \Gamma(\Delta u-pu)_{,i}dA=\int_R \Gamma(pu)_{,i}+\oint_C\left(u_{,i}\frac{\partial\Gamma}{\partial n}-\Gamma\frac{\partial u_{,i}}{\partial n}\right)dS. \quad (3.3)$$

We shall consider $\chi_i(P)$, since $\chi_i(P)-u_{,i}(P)$ is assumed known. Making use of Schwarz's inequality for vectors it follows that

$$\chi_i(P)\chi_i(P)\leq K_\Gamma\left(\frac{1}{2}\int_R pu^2dA+D(u,u)+\oint_C u_{,i}u_{,i}dS+\oint_C\frac{\partial u_{,i}}{\partial n}\frac{\partial u_{,i}}{\partial n}dS\right) \quad (3.4)$$

where $D(u,u)$ is Dirichlet integral, and

$$K_\Gamma=8\int_R \Gamma^2|\mathrm{grad}\sqrt{p}|^2dA+\int_R\Gamma^2p^2dA+\oint_C\left(\frac{\partial\Gamma}{\partial n}\right)^2dS+\oint_C\Gamma^2dS. \quad (3.5)$$

Now we have

$$\oint_C\frac{\partial u_{,i}}{\partial n}\frac{\partial u_{,i}}{\partial n}dS\leq\oint_C u_{,i}u_{,i}dS \quad (3.6)$$

and in terms of the normal coordinate system introduced in section 2 (with the boundary coordinate taken to be arc length, i.e., $y^1=S$, $y^2=N$),

$$\oint_C u_{,i}u_{,i}dS=\oint_C u^2_{|N}+2u^2_{|NS}+u^2_{|S}dS. \quad (3.7)$$

Now on C, $\Delta u-pu=u_{|NN}+u_{|SS}-pu$, so that

$$\oint_C u_{,i}u_{,i}dS=\oint_C|pu-u_{|SS}-(\Delta u-pu)|^2+u^2_{|SS}dS+2\oint_C u^2_{|NS}dS. \quad (3.8)$$

Noting that

$$u_{|SS}=\frac{\partial^2u}{\partial s^2}-K\frac{\partial u}{\partial N}=\frac{\partial^2u}{\partial s^2}+K\frac{\partial u}{\partial n} \quad (3.9)$$

we have that

$$\oint_C u_{,i}u_{,i}dS\leq 2\oint_C\left[\left(pu-\frac{\partial^2u}{\partial s^2}-|\Delta u-pu|\right)^2+\left(\frac{\partial^2u}{\partial s^2}\right)^2\right]dS+4K_M^2\oint_C\left(\frac{\partial u}{\partial n}\right)^2dS+2\oint_C u^2_{|NS}dS. \quad (3.10)$$

It follows from Green's Identity and the arithmetic-geometric mean inequality

$$2\alpha a\frac{b}{\alpha}\leq\alpha^2a^2+\frac{b^2}{\alpha^2},\qquad \alpha>0 \quad (3.11)$$

that

$$\frac{1}{2}\int_R pu^2dA+D(u,u)\leq\oint_C u\frac{\partial u}{\partial n}dS+\frac{1}{2}\int_R\frac{(\Delta u-pu)^2}{p}dA. \quad (3.12)$$

Combining (3.10), (3.12) with (3.4) we obtain

$$\chi_i(p)\chi_i(p)\leq K_\Gamma\left(\frac{1}{2}\int_R\frac{(\Delta u-pu)^2}{p}dA+2\oint_C\left[\left(pu-\frac{\partial^2u}{\partial s^2}\right)^2+\left(\frac{\partial^2u}{\partial s^2}\right)^2\right]dS+\oint_C\left(\frac{\partial u}{\partial s}\right)^2dS+\frac{1}{2}\oint_C u^2dS+\frac{11}{2}\oint_C\left(\frac{\partial u}{\partial n}\right)^2dS+2\oint_C u^2_{|NS}dS\right). \quad (3.13)$$

All terms on the right hand side of (3.13) except the last two are in terms of data. In order to bound the integral of the square of the normal derivative around the boundary we write the identity

$$\oint_C\left[f^kn_k\left(\left(\frac{\partial u}{\partial s}\right)^2-\left(\frac{\partial u}{\partial n}\right)^2\right)-2f^is_i\frac{\partial u}{\partial n}\frac{\partial u}{\partial s}\right]dS=\int_R|f^k{}_{,k}\delta^{ij}-f^i_{,j}-f^j_{,i}|u_{,i}u_{,j}dA-2\int_R f^iu_{,i}\Delta u dA, \quad (3.14)$$

which has its left hand side essentially in the form of (2.18). Here $a^{ij}=\delta^{ij}$, n_i and s_i are the unit normal and unit tangent vectors respectively and f^i is

an arbitrary continuous, piecewise continuously differentiable vector field in $R+C$. We choose f^k such that $f^k n_k > 0$ on C. (For example, if R is star-shaped with respect to the origin, then we may take $f^k = x^k$.) From (3.14) we obtain the inequality

$$\begin{aligned}\frac{3}{4}(f^k n_k)_m \oint_C \left(\frac{\partial u}{\partial n}\right)^2 dS \\ \leq \oint_C \left[(f^k n_k) + \frac{4(f^k s_k)^2}{(f^k n_k)}\right]\left(\frac{\partial u}{\partial s}\right)^2 dS \\ + (C_1 + K_1 + K_2) D(u,u) + \frac{(f^i f^j p)_M}{K_1}\int_R p u^2 dA \\ + \frac{1}{K_2}\int_R f^i f^j (\Delta u - pu)^2 dA \qquad (3.15)\end{aligned}$$

where C_1 is any bound for the largest eigenvalue of the coefficient matrix $[-f^k_{,k}\delta^{ij} + f^i_{,j} + f^j_{,i}]$, and K_1 and K_2 are positive. Let $c = \max [C_1, (f^i f^j p)_M]$ and choose $c + K_1 + K_2 = \frac{2c}{k_1}$. (For example, take $K_1 = \frac{-c + \sqrt{c^2 + 4c}}{2}$ and $K_2 = \frac{c}{K_1}$.) Then we have, using (3.12) and (3.11) with (3.15),

$$\begin{aligned}\frac{1}{2}(f^k n_k)_m \oint_C \left(\frac{\partial u}{\partial n}\right)^2 dS \leq \oint_C \left[f^k n_k \right. \\ \left. + \frac{4(f^j s_j)^2}{(f^i n_i)}\right]\left(\frac{\partial u}{\partial s}\right)^2 dS + \frac{K_1}{c}\int_R f^i f^j (\Delta u - pu)^2 dA \\ + \frac{c}{K_1}\int_R \frac{(\Delta u - pu)^2}{p} dA + \frac{4c^2}{K_1^2}\oint_C \frac{u^2}{(f^i n_i)} dS, \qquad (3.16)\end{aligned}$$

or for simplicity

$$\begin{aligned}\oint_C \left(\frac{\partial u}{\partial n}\right)^2 dS \leq a_1 \oint_C u^2 dS + a_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS \\ + a_3 \oint_C (\Delta u - pu)^2 dS \qquad (3.17)\end{aligned}$$

where

$$a_1 = \frac{8c^2}{K^2 (f^k n_k)^2_m}$$

$$a_2 = 2(f^k n_k)^{-1}_m \left[f^k n_k + \frac{f^j s_j}{(f^j n_j)_m}\right]_M$$

$$a_3 = \frac{2}{(f^k n_k)_m}\left\{\frac{K_1}{c}(f^i f^j)_M + \frac{c}{K_1}\frac{1}{p_m}\right\}$$

Finally, we need to obtain a bound for $\oint_C u^2_{|NS} dS$. We make use of (2.23) and (2.6) assuming that the vector field f^i and the tensor field A^{ij} have been chosen as indicated. We then obtain the identity

$$\begin{aligned}\oint_C (u^2_{|N} - u^2_{|S}) dS = -2\int_R f^m A^{ij} u_{|jm} (\Delta u)_{|i} dA \\ + \int_R \left[(f^m A^{ij})_{|m} g^{kl} - 2(f^i A^{lj})_{|k}\right] u_{|ik} u_{|jl} dA. \qquad (3.18)\end{aligned}$$

We assume also that f^i and A^{ij} are bounded together with their derivatives in $R+C$. (If the boundary is sufficiently smooth one can define these tensors (in the (S,N) system) as follows. Let

$$\left.\begin{aligned} &f^1 = 0, \quad f^2 = (1 - \overline{K}N) \\ &A^{22} = A^{12} = A^{21} = 0, \quad A^{11} = (1 - \overline{K}N)\end{aligned}\right\} \text{for } N \leq \frac{1}{\overline{K}}$$

$$f^i \equiv 0, \quad A^{ij} \equiv 0 \qquad \text{for } N > \frac{1}{\overline{K}}$$

where $\overline{K}$ is a constant greater than the maximum curvature K_M of C.

Using (3.11) and Schwarz's inequality we obtain from (3.18)

$$\begin{aligned}\oint_C u^2_{|NS} dS \leq \oint_C u^2_{|SS} dS + \int_R (\Delta u - pu)_{,i}(\Delta u - pu)_{,j} dA \\ + \int_R (pu)_{,i}(pu)_{,j} dA + \int_R C^{ijkl} u_{|ik} u_{|jl} dA \qquad (3.19)\end{aligned}$$

where

$$C^{ijkl} = 2f^k f^l A^{mi} A^{nj} g_{mn} + (f^m A^{ij})_{|m} g^{kl} - 2(f^l A^{ij})_{|k}. \qquad (3.20)$$

The last term on the right hand side of (3.19) is bounded as follows:

$$\int_R C^{ijkl} u_{|ik} u_{|jl} dA \leq B \int_R u_{,ij} u_{,ij} dA \qquad (3.21)$$

where B is a constant. After a moderate calculation, making use of the definitions of f^i and A^{ij} in the normal coordinate system, it is possible to obtain the bound

$$B = 1 + \overline{K} - K_m + \max\left(\overline{K}, 2|2(K')^2/\overline{K^3} - K|_M\right)$$

where K' is the derivative of the curvature with respect to arc length. Now using the divergence theorem we have

$$\begin{aligned}\int_R u_{,ij} u_{,ij} dA = \oint_C u_{,i}\left(n_j \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_j}\right) u_{,j} dS \\ + \int_R (\Delta u)^2 dA \qquad (3.22)\end{aligned}$$

where the n_i are Cartesian components of the exterior unit normal. We may rewrite the boundary integral in terms of normal and tangential derivatives and obtain

$$\begin{aligned}\int_R u_{,ij} u_{,ij} dA = +2\oint_C \frac{\partial u}{\partial n}\frac{\partial^2 u}{\partial s^2} dS \\ + \oint_C K\left(\left(\frac{\partial u}{\partial n}\right)^2 + \left(\frac{\partial u}{\partial s}\right)^2\right) dS + \int_R (\Delta u)^2 dA. \qquad (3.23)\end{aligned}$$

Using (3.11) we have

$$\int_R u_{,ij}u_{,ij}dA \leq \oint_C K\left(\frac{\partial u}{\partial s}\right)^2 dS + \oint_C \left(\frac{\partial^2 u}{\partial s^2}\right)^2 dS + \oint_C (1+K)\left(\frac{\partial u}{\partial n}\right)^2 dS + 2\int_R (\Delta u - pu)^2 dA + 2\int_R (pu)^2 dA. \quad (3.24)$$

Using (3.11) and (3.12) we have the inequality

$$\int_R (pu)^2 dA \leq p_M \left\{ \oint_C u^2 dS + \oint_C \left(\frac{\partial u}{\partial n}\right)^2 dS + \int_R \frac{(\Delta u - pu)^2}{p} dA \right\}. \quad (3.25)$$

Combining (3.17), (3.24) and (3.25) we have

$$\int_R u_{,ij}u_{,ij}dA \leq b_1 \oint_C u^2 dS + b_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS + b_3 \oint_C \left(\frac{\partial^2 u}{\partial s^2}\right)^2 dS + b_4 \int_R (\Delta u - pu)^2 dA$$

where

$$b_1 = [(1+K)_M + 2p_M]a_1 + 2p_M,\ b_2 = [(1+K)_M + 2p_M]a_2 + K_M,\ b_3 = 1 \text{ and } b_4 = 2 + [(1+K)_M + 2p_M]a_3 + 2p_M/p_m \quad (3.26)$$

Now in (3.19) we are left to consider the terms

$$\oint_C u_{|SS}^2 dS \text{ and } \int_R (pu)_{,i}(pu)_{,i}dA.$$

In order to bound the latter we can write

$$\int_R (pu)_{,i}(pu)_{,i}dA = \int_R (p_{,i}p_{,i}u^2 + 2pu_{,i}p_{,i}u + p^2 u_{,i}u_{,i})dA \leq \int_R \left(\frac{2p_{,i}p_{,i}}{p} + p^2\right)\left(\frac{1}{2}pu^2 + u_{,i}u_{,i}\right)dA \quad (3.27)$$

where (3.11) has been used with $\alpha = \sqrt{\frac{2p_{,i}p_{,i}}{p}}$.

Now (3.27) and (3.12) yield

$$\int_R (pu)_{,i}(pu)_{,i}dA \leq \frac{1}{2}\left(\frac{2p_{,i}p_{,i}}{p} + p^2\right)_M \left\{ \oint_C u^2 dS + \oint_C \left(\frac{\partial u}{\partial n}\right)^2 dS + \int_R \frac{(\Delta u - pu)^2}{p} dA \right\} \quad (3.28)$$

and hence, using (3.17) we have

$$\int_R (pu)_{,i}(pu)_{,i}dA \leq c_1 \oint_C u^2 dS + c_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS + c_3 \oint_C (\Delta u - pu)^2 dS \quad (3.29)$$

where

$$c_1 = \frac{1}{2}\left(\frac{2p_{,i}p_{,i}}{p} + p^2\right)_M (1+a_1),$$

$$c_2 = \frac{1}{2}\left(\frac{2p_{,i}p_{,i}}{p} + p^2\right)_M a_2$$

and

$$c_3 = \frac{1}{2}\left(\frac{2p_{,i}p_{,i}}{p} + p^2\right)_M \left(a_3 + \frac{1}{p_M}\right).$$

Finally we have from (3.9), (3.11), and (3.17)

$$\oint_C u_{|SS}^2 dS \leq 2K_M^2 a_1 \oint_C u^2 dS + 2K_M^2 a_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS + 2\oint_C \left(\frac{\partial^2 u}{\partial s^2}\right)^2 dS + 2K_M^2 a_3 \int_R (\Delta u - pu)^2 dA. \quad (3.30)$$

Combining (3.19), (3.21), (3.26), (3.29) and (3.30) we have

$$\oint_C u_{|NS}^2 dS \leq d_1 \oint_C u^2 dS + d_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS + d_3 \oint_C \left(\frac{\partial^2 u}{\partial s^2}\right)^2 dS + d_4 \int_R (\Delta u - pu)^2 dA + d_5 \int_R (\Delta u - pu)_{,i}(\Delta u - pu)_{,i}dA \quad (3.31)$$

where

$$d_1 = 2K_M^2 a_1 + Bb_1 + c_1,\ d_2 = 2K_M^2 a_2 + Bb_2 + c_2,$$

$$d_3 = 2 + Bb_3,\ d_4 = 2K_M^2 a_3 + Bb_4 + c_3 \text{ and } d_5 = 1.$$

Now (3.17) and (3.31) may be inserted into (3.13) to yield the desired bound,

$$\chi_i\chi_i(P) \leq K_1 \left\{ \beta_1 \oint_C u^2 dS + \beta_2 \oint_C \left(\frac{\partial u}{\partial s}\right)^2 dS + \beta_3 \oint_C \left(\frac{\partial^2 u}{\partial s^2}\right)^2 dS + \beta_5 \int_R (\Delta u - pu)^2 dA + \beta_4 \int_R (\Delta u - pu)_{,i}(\Delta u - pu)_{,i}dA \right\} \quad (3.32)$$

where the β_j's are explicitly determined constants.

It should be pointed out that for the sake of simplicity, no effort was made to obtain the "best" such bound. Better bounds could be obtained, for example, by leaving certain known quantities under the sign of integration (compare equations (3.16) and (3.17)) rather than replacing them by a maximum value.

4. Higher Derivatives in N Dimensions

The process described in section 3 can be generalized easily to higher derivatives. In bounding the Mth derivative we must require the existence and integrability of Mth derivatives of $\Delta u - pu$, pu, and $M+1$ tangential derivatives of u on C. We also assume that bounds have been obtained for the various lower order differential invariants.

The mode of generalization will become apparent from the case $M=2$ where we assume that bounds have been obtained for

$$\int_R pu^2 dv, D(u,u), \int_R u_{,ij}u_{,ij}dv, \oint_C \frac{\partial u}{\partial n} dS, \oint_C u_{,ij}u_{,ij}dS \tag{4.1}$$

as was done explicitly in two dimensions in the previous section.

If ω_n denotes the surface area of the unit sphere in N dimensions then

$$\Gamma = \frac{-1}{(N-2)\omega_n r_{PQ}^{N-2}}, \quad N>2 \tag{4.2}$$

is a fundamental solution of $\Delta u=0$. We can write

$$\chi_{ij}(P) \equiv u_{,ij}(P) - \int_R \Gamma(\Delta u - pu)_{,ij}dv = \int_R \Gamma(pu)_{,ij}dv + \oint_C \left(u_{,ij}\frac{\partial \Gamma}{\partial n} - \Gamma \frac{\partial u_{,ij}}{\partial n}\right) dS. \tag{4.3}$$

Since $\int_R \Gamma^2 dv$ does not exist for $N \geq 4$ we cannot apply Schwarz's inequality, as was done in (3.4), for dimensions higher than three.

For $N \geq 4$ we apply Schwarz's inequality as follows

$$\left(\int_R \Gamma u dv\right)^2 \leq \int_R r^{N-1}\Gamma^2 dv \int_R r^{-(N-1)}u^2 dv$$

where the singularity of Γ is chosen as the origin. In order to bound $\int_R r^{-(N-1)}u^2 dv$ we use "Green's Identity"

$$-\int_R [\Delta(r^{-(N-3)})u^2 - r^{-(N-3)}\Delta u^2]dv = \text{Boundary Integrals.}$$

Now using (3.11) we can easily obtain

$$\int_R r^{-(N-1)}u^2 dv \leq \frac{4}{2N-7}\int_R r^{-(N-3)}(\Delta u - pu)^2 dv + \text{Boundary Integrals.}$$

This same basic technique may be used to handle all terms in (4.3) for $N \geq 4$.

The boundary integrals occurring in the above inequalities are the same as those to be dealt with below.

We shall show how the desired bounds may be obtained for $N \leq 3$ since in higher dimensions only technical details enter.

Applying Schwarz's inequality, as was done in (3.4), we see that

$$\chi_{ij}(P)\chi_{ij}(P) \leq K_\Gamma \left\{ \int_R (pu)_{,ij}(pu)_{,ij}dv + \oint_C u_{,ij}u_{,ij}dS + \oint_C \frac{\partial u_{,ij}}{\partial n}\frac{\partial u_{,ij}}{\partial n}dS \right\} \tag{4.4}$$

where K_Γ is given by (3.5).

We see that

$$\int_R (pu)_{,ij}(pu)_{,ij}dv \leq 3\left(\frac{p_{,ij}p_{,ij}}{p}\right)_M \int_R pu^2 + 12(p_{,i}p_{,i})_M D(u,u) + 3p_M^2 \int_R u_{,ij}u_{,ij}dv, \tag{4.5}$$

but by assumption each integral on the right side of (4.5) has a known bound. Similarly we assume that a bound is known for $\oint_C u_{,ij}u_{,ij}dS$.

It now remains for us to obtain bounds for the last term of (4.4). We see that

$$\oint_C \frac{\partial u_{,ij}}{\partial n}\frac{\partial u_{,ij}}{\partial n}ds = \oint_C g^{ij}g^{kl}\overline{u}_{|ikN}\overline{u}_{|jlN}dS = \oint_C g^{\alpha\beta}g^{\gamma\delta}\overline{u}_{|\alpha\gamma N}\overline{u}_{\beta\delta|N}dS + 2\oint_C g^{\alpha\beta}\overline{u}_{|\alpha NN}\overline{u}_{|\beta NN}dS + \oint_C (\overline{u}_{|NNN})^2 dS. \tag{4.6}$$

From (2.15) we see that

$$\overline{u}_{|\sigma NN} = (\Delta\overline{u} - \overline{p}\overline{u})_{|\sigma} + (\overline{p}\overline{u})_{|\sigma} - g^{\alpha\beta}\overline{u}_{|\alpha\beta\sigma}. \tag{4.7}$$

The first two expressions on the right are known and

$$\overline{u}_{|\alpha\beta\gamma} = \frac{\partial}{\partial y^\gamma}\left[\frac{\partial^2\overline{u}}{\partial y^\alpha \partial y^\beta} - \left\{\begin{matrix} k \\ \alpha\beta \end{matrix}\right\}\overline{u}_{|k}\right] - \left\{\begin{matrix} k \\ \alpha\gamma \end{matrix}\right\}\overline{u}_{|k\beta} - \left\{\begin{matrix} k \\ \beta\gamma \end{matrix}\right\}\overline{u}_{|\alpha k}, \tag{4.8}$$

where

$$\frac{\partial \overline{u}_{|k}}{\partial y^\gamma} = \overline{u}_{|\gamma k} + \left\{\begin{matrix} j \\ \gamma k \end{matrix}\right\}\overline{u}_{|j}. \tag{4.9}$$

The first term on the right side of (4.8) is known. We can use the inequalities

$$b^{ij}\overline{u}_{|i}\overline{u}_{|j} \leq \tau_1 g^{ij}\overline{u}_{|i}\overline{u}_{|j},$$
$$b^{ijkl}\overline{u}_{|ij}\overline{u}_{|kl} \leq \tau_2 g^{ij}g^{kl}\overline{u}_{|ik}\overline{u}_{|jl}, \tag{4.10}$$

where τ_1, τ_2 are upper bounds for the largest eigenvalue in each case, to complete the bound for

$$\oint_C g^{\alpha\beta}\overline{u}_{|\alpha NN}\overline{u}_{|\beta NN}dS$$

by a function of the quantities (4.1).

Furthermore, from (2.15) we see that

$$\overline{u}_{|NNN}=(\Delta\overline{u}-\overline{pu})_{|N}+(\overline{pu})_{|N}-g^{\alpha\beta}\overline{u}_{|\alpha\beta N} \quad (4.11)$$

and hence we can bound the last term of (4.6) in terms of known quantities and a bound for

$$\oint_C g^{\alpha\beta}g^{\gamma\delta}\overline{u}_{|\alpha\gamma N}\overline{u}_{|\beta\delta N}dS.$$

From (2.7) for $M=2$ and the choice

$$\begin{aligned}&\overline{f}^N=1, \qquad \overline{f}^\alpha=0,\\ &\overline{A}^{\alpha\beta}=g^{\alpha\beta}, \qquad \overline{A}^{i\alpha}=\overline{A}^{NN}=0\\ &\overline{B}^{NN}=1, \qquad \overline{B}^{ij}=0 \qquad i,j\neq N,\end{aligned} \quad (4.12)$$

we have

$$\begin{aligned}\oint_C &[g^{\alpha\beta}g^{\gamma\delta}\overline{u}_{|\alpha\gamma N}\overline{u}_{|\beta\delta N}-g^{\alpha\beta}\overline{u}_{|\alpha NN}\overline{u}_{|\beta NN}]dS\\ &=-\int_R f^m A^{ij}B^{kl}[(\Delta u-pu)+pu]_{|ik}u_{|jlm}dv\\ &+\int_R\{(f^mA^{ij}B^{kl}\delta^{rs})_{|m}-2(f^sA^{ij}B^{kl})_{|r}\}u_{|iks}u_{|jlr}dV.\end{aligned} \quad (4.13)$$

Hence

$$\begin{aligned}\oint_C g^{\alpha\beta}g^{\gamma\delta}\overline{u}_{|\alpha\gamma N}\overline{u}_{|\beta\delta N}dS &\leq \oint_C g^{\alpha\beta}\overline{u}_{|\alpha NN}\overline{u}_{|\beta NN}dS\\ &+\int_R A^{ij}B^{kl}[(\Delta u-pu)_{|ik}(\Delta u-pu)_{|jl}\\ &+(pu)_{|ik}(pu)_{|jl}]dv+I(u_{|ijk}),\end{aligned} \quad (4.14)$$

where

$$\begin{aligned}I(u_{|ijk})=\int_R &\{(f^mA^{ij}B^{kl}\delta^{rs})_{|m}-2(f^sA^{ij}B^{kl})_{|r}\\ &+f^rf^sA^{ij}B^{kl}\}u_{|ikr}u_{|jls}dv.\end{aligned} \quad (4.15)$$

But

$$\begin{aligned}I(u_{|ijk})&\leq\tau_3\int_R u_{,ijk}u_{,ijk}dv\\ &=\tau_3\left\{\oint_C u_{,ij}u_{,ijk}\nu_k ds-\int_R(\Delta u)_{,ij}u_{,ij}dv\right\}\\ &=\frac{(\tau_3)^2}{4\epsilon}\oint_C g^{ij}g^{kl}\overline{u}_{|ik}\overline{u}_{|jl}ds-\tau_3\int_R(\Delta u)_{,ij}u_{,ij}dv\\ &+\epsilon\oint_C g^{ij}g^{kl}\overline{u}_{|ikN}\overline{u}_{|jlN}ds\end{aligned} \quad (4.16)$$

where τ_3 is again an upper bound for the largest eigenvalue of the coefficient matrix in (4.15). We can substitute (4.1) into (4.4) for $\epsilon<1$ and achieve the desired bound for

$$\oint_C g^{\alpha\beta}g^{\gamma\delta}\overline{u}_{|\alpha\gamma N}\overline{u}_{|\beta\delta N}ds.$$

5. References

[1] S. Bergman and M. Schiffer, Kernel functions and elliptic differential equations in mathematical physics, Academic Press, New York (1953).

[2] J. H. Bramble and L. E. Payne, Bounds in the first biharmonic boundary value problem (to be published).

[3] J. B. Diaz, Upper and lower bounds for quadratic integrals, and at a point, for solutions of linear boundary value problems, Proc. Symp. Boundary Val. Probs. Differential Equations, U.S. Army Research Center, Univ. of Wisconsin, Apr. 20–22, 1959.

[4] L. Hörmander, Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order, Compt. Rend. Douzième Congr. Mathématiciens Scandinaves Tenu à Lund, pp. 105–115 (1953).

[5] B. E. Hubbard, Bounds for eigenvalues of free and fixed membrane by finite difference methods, Pac. J. Math. **11**, 559–590 (1961).

[6] L. E. Payne and H. F. Weinberger, New bounds for solutions of second order elliptic partial differential equations, Pac. J. Math. **8**, No. 3, 551–573 (1958).

[7] F. Rellich, Darstellung der Eigenwerte von $\Delta u+\lambda u$ durch ein Randintegral, Math. Z. **46**, 635–636 (1940).

[8] J. L. Synge and A. Schild, Tensor calculus (Univ. of Toronto Press, Toronto, 1952).

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A Priori Bounds in the First Boundary Value Problem in Elasticity

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In this paper we obtain bounds for the strain energy, and pointwise bounds for the solution of the first boundary value problem in the equations of elasticity. The inequalities derived are such that the Rayleigh-Ritz technique may be used to improve the bounds.

1. Introduction

In a recent paper [2] [2] the authors gave a method for obtaining bounds in the Dirichlet problem for second-order elliptic partial differential equations. Pointwise bounds were obtained which involved, in addition to the approximating function, only undifferentiated data of the problem. The inequalities were of a priori type, hence only regularity conditions were imposed on the approximating functions.

In this paper we obtain bounds for the strain energy, and pointwise bounds for the solution of the first boundary value problem in the equations of elasticity. Other methods for obtaining bounds, which are based on known variational principles, have been given in the literature (see, for instance, Prager and Synge [9], Diaz and Greenberg [3], Synge [10], and others).

A method based on a priori inequalities was proposed by Payne and Weinberger [8]. Their method involved the introduction of a somewhat complicated unit-tangent vector, and pointwise bounds involved differentiated data. We derive a method which yields pointwise bounds which involve the undifferentiated data of the problem.

In the problem under consideration the displacement vector with components u_i satisfies the following system of second-order equations in some N-dimensional bounded domain D,

$$L_i(u)=u_{i,jj}+\alpha u_{j,ji}=F_i \tag{1.1}$$

where α involves the elastic constants λ and μ ($\alpha=(\lambda+\mu)/\mu=(1-2\sigma)^{-1}$, σ denoting Poisson's ratio), and F_i is proportional to the body force. In (1.1) $_{,ji}$ denotes $\partial^2/\partial x_j\partial x_i$ and a repeated index indicates summation from 1 to N, where N denotes the number of dimensions. By specializing to two or three dimensions we can treat the two physically interesting cases. In addition to the differential equation (1.1) the displacement components satisfy the condition

$$u_i=f_i \tag{1.2}$$

on the boundary C of D. Regions which extend to infinity may also be treated if suitable conditions are imposed at infinity.

The strain energy $E(u,u)$ for problem (1.1), (1.2) is defined as:

$$2\mu^{-1}E(u,u)=\int_D\left[\frac{1}{2}(u_{i,j}+u_{j,i})(u_{i,j}+u_{j,i})+(\alpha-1)(u_{j,j})^2\right]dV. \tag{1.3}$$

Our problem then is that of obtaining upper and lower bounds for $E(u,u)$ and for the value of $u_i(P)$ at a point P in D. As we shall see, these bounds follow immediately once we have established certain a priori bounds. Let us suppose that we are able to establish the following inequality for sufficiently smooth functions w_i:

$$E(w,w)\leq K_1\oint_C \eta_1 w_i w_i dS+K_2\oint_C \eta_2\,\mathrm{grad}_s w_i\cdot\mathrm{grad}_s w_i dS+K_3\int_D L_i(w)L_i(w)dV \tag{1.4}$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

where η_1 and η_2 are given positive functions on C, and K_1, K_2, and K_3 are explicitly determined. Here grad_s denotes the tangential projection of the gradient. Now if we set

$$w_i = u_i - \phi_i, \tag{1.5}$$

the right hand side of (1.4) will involve only the data of (1.1) and (1.2) and the approximating function ϕ_i. We would then have an a priori bound for $E(u-\phi,u-\phi)$ which could be made small if the functions ϕ_i could be chosen to approximate the data closely in the mean square sense. But by the triangle inequality

$$\sqrt{E(\phi,\phi)} - \sqrt{E(u-\phi,u-\phi)} \leq \sqrt{E(u,u)} \leq \sqrt{E(\phi,\phi)} + \sqrt{E(u-\phi,u-\phi)}. \tag{1.6}$$

Hence knowledge of the a priori inequality (1.4) would give upper and lower bounds for $E(u,u)$. The inequality (1.4) with $w_i = u_i - \phi_i$ is suitable for application of the Rayleigh-Ritz technique for improving the bounds.

The pointwise bounds will also follow from appropriate a priori inequalities. We concern ourselves then in this paper with the establishment of the necessary a priori inequalities.

2. Auxiliary Inequalities

In this section we employ some generalized Green's identities to derive certain inequalities which will be used in sec. 3. We consider first a function h harmonic in D. As indicated in [8, eq 2.4] the function h satisfies the following identity:

$$\oint_C [f^k n_k h_{,i} h_{,i} - 2 f^i n_j h_{,i} h_{,j}] dS = \int_D [f^k_{,k} \delta_{ij} - 2 f^i_{,j}] h_{,i} h_{,j} dV \tag{2.1}$$

where the f^k are components of a piecewise continuous vector field and δ_{ij} denotes the Kronecker delta. The n_j denote the components of the unit normal directed outward from D on C. In [8, eq 2.4] we have taken $a^{ij} \equiv \delta_{ij}$ and used the fact that h is harmonic. By breaking up the derivatives of h in the boundary integral into normal and tangential components we obtain

$$\oint_C f^k n_k \left[|\text{grad}_s h|^2 - \left(\frac{\partial h}{\partial n}\right)^2 \right] dS = \int_D [f^k_{,k} \delta_{ij} - 2 f^i_{,j}] h_{,i} h_{,j} dV + 2 \oint_C f^i \, \text{grad}_s h \cdot \text{grad}_s x_i \frac{\partial h}{\partial n} dS \tag{2.2}$$

where $\overline{\dfrac{\partial h}{\partial n}}$ denotes the normal derivative of h on C. If we now choose the f^k in such a way that $t = f^k n_k > 0$ on C, we obtain the inequality

$$\oint_C t \left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t \, |\text{grad}_s h|^2 dS - 2 \oint_C f^i \, \text{grad}_s h \cdot \text{grad}_s x_i \frac{\partial h}{\partial n} dS + aD(h,h). \tag{2.3}$$

In (2.3) we have used the notation

$$D(h,h) = \int_D h_{,i} h_{,i} dV \tag{2.4}$$

for the Dirichlet integral, and the inequality

$$\int_D [2 f^i_{,j} - f^k_{,k} \delta_{ij}] h_{,i} h_{,j} dV \leq aD(h,h). \tag{2.5}$$

The constant a is clearly any upper bound for the largest eigenvalue of the coefficient matrix $2f^i_{,j} - f^k_{,k}\delta_{ij}$. Then from Green's first identity, i.e.,

$$D(h,h) = \oint_C h \frac{\partial h}{\partial n} dS, \tag{2.6}$$

we obtain from (2.3)

$$\oint_C t\left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t\,|\text{grad}_s\, h|^2 dS - \oint_C [2f^i\,\text{grad}_s\, h\cdot\text{grad}_s\, x_i - ah]\,\frac{\partial h}{\partial n}\,dS. \tag{2.7}$$

Now from Schwarz's inequality we have

$$\oint_C t\left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t\,|\text{grad}_s\, h|^2 dS + \left\{\oint_C t^{-1}[2f^i\,\text{grad}_s\, h\cdot\text{grad}_s\, x_i - ah]^2 dS\cdot\oint_C t\left(\frac{\partial h}{\partial n}\right)^2 dS\right\}^{1/2}, \tag{2.8}$$

from which we obtain, by completing the square,

$$\left\{\oint_C t\left(\frac{\partial h}{\partial n}\right)^2 dS\right\}^{1/2} \leq \frac{1}{2}\left\{\oint_C t^{-1}[2f^i\,\text{grad}_s\, h\cdot\text{grad}_s\, x_i - ah]^2 dS\right\}^{1/2}$$
$$+\left\{\oint_C t\,|\text{grad}_s\, h|^2 dS + \frac{1}{4}\oint_C t^{-1}[2f^i\,\text{grad}_s\, h\cdot\text{grad}_s\, x_i - ah]^2 dS\right\}^{1/2} \tag{2.9}$$

or, using the arithmetic-geometric mean inequality, the cruder inequality

$$\oint_C t\left(\frac{\partial h}{\partial n}\right)^2 dS \leq \oint_C t^{-1}[2f^i\,\text{grad}_s\, h\cdot\text{grad}_s\, x_i - ah]^2 dS + 2\oint_C t\,|\text{grad}_s\, h|^2 dS. \tag{2.10}$$

This inequality bounds the normal derivative of a harmonic function h in terms of h and its tangential derivatives on C. (Instead of (2.10) the somewhat better though more involved inequality [8, eq 3.17], with $a^{ij}=\delta_{ij}$, could be used.) In deriving (2.10) we have used the procedures employed in [8]. For completeness and because the case of the harmonic function was not given explicitly in the previous paper we have again presented the derivation of (2.10).

Now let ψ_i be the ith component of a vector function which possesses piecewise continuous second derivatives in D and vanishes on C. Then consider the following identity:

$$2\int_D f^k\psi_{i,k}L_i(\psi)\,dV = 2\oint_C f^k\psi_{i,k}[\psi_{i,j}n_j + \alpha\psi_{j,j}n_i]\,dS + \int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dV$$
$$-2\int_D [f^k_{,j}\psi_{i,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}]\,dV - \oint_C f^k n_k[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dS. \tag{2.11}$$

Since ψ_i vanishes on C the first and last integrals on the right combine to give (upon rearranging of (2.11))

$$\oint_C f^k n_k[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dS = 2\int_D f^k\psi_{i,k}L_i(\psi)\,dV$$
$$-\int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dV + 2\int_D [f^k_{,j}\psi_{j,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}]\,dV. \tag{2.12}$$

Since f^i has bounded first derivatives we may easily obtain, for α positive,

$$2\int_D [f^k_{,j}\psi_{i,k}\psi_{i,j} + \alpha f^k_{,i}\psi_{i,k}\psi_{j,j}]\,dV - \int_D f^k_{,k}[\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dV \leq b\int_D [\psi_{i,j}\psi_{i,j} + \alpha\psi_{i,i}\psi_{j,j}]\,dV, \tag{2.13}$$

where the constant b may be easily determined. (The physically interesting values of α lie in the interval $(1/3,\infty)$.)

Note that for ψ_i vanishing on C the integral on the right of (2.13) is proportional to the strain energy $E(\psi,\psi)$. Also the quantity inside the square bracket on the left of (2.12) is proportional to the sum of the squares of the surface tractions. By the arithmetic-

geometric mean inequality and the fact that α is positive, we have then

$$\oint_C t[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dS\leq\left[b+\frac{[f^kf^k]_{\max}}{\beta}\right]\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV+\beta\int_D L_i(\psi)L_i(\psi)dV \tag{2.14}$$

for any positive β. Now since ψ_i vanishes on C we have

$$\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV=-\int_D\psi_iL_i(\psi)\,dV \tag{2.15}$$

or by Schwarz's inequality

$$\frac{\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}{\int_D\psi_i\psi_i dV}\leq\frac{\int_D L_i(\psi)L_i(\psi)dV}{\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}. \tag{2.16}$$

However, for $\alpha>0$

$$\frac{\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV}{\int_D\psi_i\psi_i dV}\geq\frac{\int_D\psi_{i,j}\psi_{i,j}dV}{\int_D\psi_i\psi_i dV}\geq\lambda_1, \tag{2.17}$$

where λ_1 is the first eigenvalue in the fixed membrane problem for D. Lower bounds for λ_1 are easily derivable from the Faber-Krahn inequality [5], [6], or from monotony principles. Thus from (2.16) and (2.17) we have

$$\int_D\psi_i\psi_i dV\leq\frac{1}{\lambda_1}\int_D[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dV\leq\frac{1}{\lambda_1^2}\int_D L_i(\psi)L_i(\psi)dV. \tag{2.18}$$

The insertion of (2.18) into (2.14) then yields

$$\oint_C t[\psi_{i,j}\psi_{i,j}+\alpha\psi_{i,i}\psi_{j,j}]dS\leq[b/\lambda_1+2\{[f^kf^k]_{\max}/\lambda_1\}^{\frac{1}{2}}]\int_D L_i(\psi)L_i(\psi)dV, \tag{2.19}$$

where we have made the optimal choice for β.

3. A Priori Inequalities

Let w_i have piecewise continuous second derivatives in D. We seek an a priori bound for $E(w,w)$. To this end we decompose w_i as

$$w_i=\overline{w}_i+w_i' \tag{3.1}$$

where

$$L_i(\overline{w})=0 \tag{3.2}$$

in D and

$$\overline{w}_i=w_i \tag{3.3}$$

on C. The function w_i' then satisfies

$$L_i(w')=L_i(w) \tag{3.4}$$

in D and

$$w'_i=0 \tag{3.5}$$

on C. Then clearly

$$E(w,w)=E(\overline{w},\overline{w})+E(w_i'w') \tag{3.6}$$

since $\overline{w}$ and w' are orthogonal in E-norm (i.e., $E(\overline{w},w')=0$). We derive first a bound for

$E(w',w')$. From the fact that $w_i'=0$ on C we have from (2.18), with ψ_i replaced by w_i',

$$2\mu^{-1}E(w',w')\leq\frac{1}{\lambda_1}\int_D L_i(w')L_i(w')dV=\frac{1}{\lambda_1}\int_D L_i(w)L_i(w)dV. \tag{3.7}$$

Just as one establishes the Dirichlet principle for harmonic functions it is possible in bounding $E(\overline{w},\overline{w})$ to show that

$$E(\overline{w},\overline{w})\leq E(H,H) \tag{3.8}$$

where H is any vector which satisfies the same boundary values as $\overline{w}$ on C. Let H_i be the components of the harmonic vector which takes on the same boundary values as $\overline{w}_i$. Then

$$\begin{aligned} E(\overline{w},\overline{w})&\leq\overline{\mu}\int_D H_{i,j}H_{i,j}dV, \qquad \overline{\mu}=\begin{cases}\mu[2+N(\alpha-1)], & \alpha>1\\ 2\mu & ,\alpha\leq 1\end{cases}\\ &=\overline{\mu}\oint_C H_i\frac{\partial}{\partial n}H_i\,dS, \end{aligned} \tag{3.9}$$

the latter equality following from Green's theorem and the fact that H_i is harmonic. Now choose f^i in such a way that $t=f^in_i>0$ on C. Then by the arithmetic-geometric mean inequality we have

$$E(\overline{w},\overline{w})\leq\frac{\overline{\mu}}{\gamma}\oint_C t^{-1}H_iH_i\,dS+\mu\gamma\oint_C t\frac{\partial H_i}{\partial n}\frac{\partial H_i}{\partial n}dS \tag{3.10}$$

for any positive γ. Now since the H_i are harmonic we may use inequality (2.10) for vectors to obtain

$$\begin{aligned}\oint_C t\frac{\partial H_i}{\partial n}\frac{\partial H_i}{\partial n}dS\leq\oint_C t^{-1}[2f^j\operatorname{grad}_s H_i\cdot\operatorname{grad}_s x_j-aH_i][2f^k\operatorname{grad}_s H_i\cdot\operatorname{grad}_s x_k-aH_i]\,dS\\ +2\oint_C t\operatorname{grad}_s H_i\cdot\operatorname{grad}_s H_i\,dS.\end{aligned} \tag{3.11}$$

Now since $H_i=\overline{w}_i=w_i$ on C we have upon combining (3.6), (3.7), (3.10), and (3.11) the desired bound

$$\begin{aligned}2\overline{\mu}^{-1}E(w,w)\leq\frac{1}{\gamma}\oint_C t^{-1}w_iw_i\,dS+\gamma\Big\{2\oint_C t\operatorname{grad}_s w_i\cdot\operatorname{grad}_s w_i\,dS\\ +\oint_C t^{-1}[2f^j\operatorname{grad}_s w_i\cdot\operatorname{grad}_s x_j-aw_i][2f^k\operatorname{grad}_s w_i\cdot\operatorname{grad}_s x_k-aw_i]\,dS\Big\}+\frac{\mu/\overline{\mu}}{\lambda_1}\int_D L_i(w)L_i(w)dV\end{aligned} \tag{3.12}$$

which can clearly be put in the form (1.4) if so desired.

We now derive a bound for the integral of the sum of the squares of the displacement components w_i over D. This inequality will be used in sec. 5 to derive pointwise bounds for w_i. We seek then a bound for $\int_D w_iw_i\,dV$.

To this end we introduce the auxiliary vector χ with components χ_i defined by

$$L_i(\chi)=w_i \tag{3.13}$$

in D and

$$\chi_i=0 \tag{3.14}$$

on C. Then

$$\int_D w_iw_i\,dV=\int_D w_iL_i(\chi)dV=\oint_C w_i[\chi_{i,j}n_j+\alpha\chi_{j,i}n_j]dS+\int_D\chi_iL_i(w)dV. \tag{3.15}$$

By Schwarz's inequality for vectors we have

$$\int_D w_i w_i dV = \int_D L_i(\chi) L_i(\chi) dV \leq \left\{ \oint_C t^{-1}[w_i w_i + (w_i n_i)^2] dS \oint_C t[\chi_{i,j}\chi_{i,j} + \alpha \chi_{i,i}\chi_{i,i}] dS \right\}^{\frac{1}{2}} + \left\{ \int_D \chi_i \chi_i dV \int_D L_i(w) L_i(w) dV \right\}^{\frac{1}{2}}. \quad (3.16)$$

Since χ_i vanishes on C we may use (2.18) and (2.19) with ψ_i replaced by χ_i to obtain

$$\left[\int_D w_i w_i dV \right]^{\frac{1}{2}} \leq \left\{ [b/\lambda_1 + 2([f^k f^k]_{\max}/\lambda_1)^{\frac{1}{2}}] \oint_C t^{-1}[w_i w_i + (w_i n_i)^2] dS \right\}^{\frac{1}{2}} + \left\{ \frac{1}{\lambda_1^2} \int_D L_i(w) L_i(w) dV \right\}^{\frac{1}{2}}. \quad (3.17)$$

If we should now set $w_i = u_i - \phi_i$ where u_i is the solution of (1.1), (1.2) and ϕ_i is an approximating function then the right hand side of (3.17) would involve u_i only through its data. We employ this inequality in the next section for obtaining pointwise bounds for w_i.

4. Pointwise Bounds

In order to obtain pointwise bounds we again decompose w_i as in eq (3.1). Pointwise bounds for $\overline{w}_i$ are then derived by making use of a mean-value theorem for the solution of the homogeneous equations of elasticity. Such a mean-value theorem was given in the three-dimensional case by Aquaro [1] and Synge [10]. The analogous N-dimensional result was obtained by Diaz and Payne [4].

Let $S(r)$ be a sphere of radius r and surface $\sum(r)$ with center at the point at which pointwise bounds are sought. We choose the origin of a spherical-coordinate system at this point, and require that $S(r)$ be contained in D.

In what follows whenever a Greek index is repeated the summation convention is not to be used, i.e., $u_{\alpha,i}u_{\alpha,i}$ will denote the square of the gradient of a single-displacement component u_α.

From the mean-value theorem [4, eq 3.9]

$$r^{N+1}\overline{w}_\beta(0) = \frac{N}{2(N+\alpha)w_N} \oint_{\sum(r)} [(2-\alpha)\delta_{\beta j} r^2 + (N+2)\alpha x_\beta x_j] \overline{w}_j dS. \quad (4.1)$$

If we now integrate (4.2) with respect to r from the origin to R, with R so chosen that $S(R)$ is still contained in D, we obtain

$$\overline{w}_\beta(0) = \frac{N(N+2)}{2(N+\alpha)R^{N+2}w_N} \int_{S(R)} [(2-\alpha)\delta_{\beta j} r^2 + (N+2)\alpha x_\beta x_j] \overline{w}_j dV. \quad (4.2)$$

An application of Schwarz's inequality then yields

$$|\overline{w}_\beta(0)|^2 \leq \frac{N}{N+4} \left[\frac{(N+2)}{2(N+\alpha)} \right]^2 \left\{ \frac{N(2-\alpha)^2 + 2\alpha(2-\alpha)(N+2) + (N+2)^2\alpha^2}{w_N R^N} \right\} \int_D \overline{w}_j \overline{w}_j dV$$

$$\equiv A(N) R^{-N} \int_D \overline{w}_j \overline{w}_j dV. \quad (4.3)$$

By our construction we see that R is actually the distance from the origin to the boundary.

Now from (3.17), since $L_i(\overline{\ })\equiv 0$ it follows that

$$|\overline{w}_\beta(0)|^2 \le A(N)K_1^2 R^{-N}\oint_C t^{-1}[\overline{w}_i\overline{w}_i+(\overline{w}_i n_i)^2]dS$$

$$=A(N)K_1^2 R^{-N}\oint_C t^{-1}[w_i w_i+(w_i n_i)^2]dS \tag{4.4}$$

where

$$K_1=[b/\lambda_1+2([f_r^k f^k]_{\max}/\lambda_1)^{1/2}]. \tag{4.5}$$

In order to obtain bounds for $w'_\beta(0)$ we introduce the Kelvin fundamental solution $z_\beta^k(O,Q)$ which denotes the components of displacement in the kth direction corresponding to a point load at the origin in the βth direction, i.e.,

$$z_\beta^k=\begin{cases}\dfrac{1}{4\pi\mu(\alpha+1)}[(\alpha+2)\log(1/r)\delta_{\beta k}+\alpha x_\beta x_k r^{-2}], & N=2\\[2ex] \dfrac{1}{2\mu(N-2)(\alpha+1)w_N}[(\alpha+2)r^{-(N-2)}\delta_{\beta k}+(N-2)\alpha x_\beta x_k r^{-N}], & N>2.\end{cases} \tag{4.6}$$

For $N=2, 3$ these expressions are given in Love [7]. By the analogue of Green's third identity for the equations of elasticity we have

$$w'_\beta(0)=\oint_C z_\beta^k[w'_{k,j}n_j+\alpha w'_{j,j}n_k]dS+\int_D z_\beta^k L_k(w')dV. \tag{4.7}$$

Since $L_k(w')\equiv L_k(w)$ in D, for the boundary-value problem under consideration $L_k(w')$ will be known. Hence by Schwarz's inequality

$$|w'_\beta(0)|^2\le\left\{\left[\oint_C t^{-1}(z_\beta^k z_\beta^k+[z_\beta^k n_k]^2)dS\oint_C t(w'_{i,j}w'_{i,j}+\alpha w'_{i,i}w'_{j,j})dS\right]^{1/2}+\left|\int_D z_\beta^k L_k(w)dV\right|\right\}^2. \tag{4.8}$$

Now since w'_i vanishes on C we may use inequality (2.19) to bound the second boundary integral on the right of (4.8); i.e.,

$$|w'_\beta(0)|^2\le\left\{\left[K_1\oint_C t^{-1}(z_\beta^k z_\beta^k+[z_\beta^k n_k]^2)dS\int_D L_j(w)L_j(w)dV\right]^{1/2}+\left|\int_D z_\beta^k L_k(w)dV\right|\right\}^2. \tag{4.9}$$

Now since

$$|w_\beta(0)|^2\le 2\,[\,|\overline{w}_\beta(0)|^2+|w'_\beta(0)|^2\,] \tag{4.10}$$

we have by inequalities (4.4) and (4.10) a bound for any one of the displacement components at the origin.

We may also obtain pointwise bounds for the derivatives of u_β at points in D. Again we first obtain an a priori bound for the derivative of a general function w_β and interpret our results for $w_\beta=u_\beta-\phi_\beta$. As before w_β is decomposed as $w_\beta=\overline{w}_\beta+w'_\beta$. The pointwise bounds for the derivatives of $\overline{w}_\beta$ are again obtained by use of mean-value inequalities while the fundamental solution (4.6) is used to obtain derivatives of w'_β. Diaz and Payne [4, eq 4.14] have shown that the stress component $\overline{\tau}_{\beta\gamma}$ defined as

$$\overline{\tau}_{\beta\gamma}=\mu\,[(\overline{w}_{\beta,\gamma}+\overline{w}_{\gamma,\beta})+(\alpha-1)\overline{w}_{k,k}\delta_{\beta\gamma}] \tag{4.11}$$

satisfies the mean value theorem

$$r^{N+2}\overline{\tau}_{\beta\gamma}(0)=\frac{\mu N}{w_N(N+2\alpha+2)}\Big[(N+2)(1-\alpha)\oint_{\Sigma(r)} r^2(\overline{w}_\beta x_\gamma+\overline{w}_\gamma x_\beta)dS+\alpha(N+2)(N+4)\oint_{\Sigma(r)} x_\beta x_\gamma x_k\overline{w}_k dS$$

$$+(2\alpha^2-4\alpha-N-2)\delta_{\beta\gamma}\oint_{\Sigma(r)} r^2\overline{w}_k x_k dS\Big]. \tag{4.12}$$

Upon integrating with respect to r as before from 0 to R we obtain

$$R^{N+4}\bar{\tau}_{\beta\gamma}(0)=\frac{\mu N(N+4)}{w_N(N+2\alpha+2)}\int_{S(R)}[(N+2)(1-\alpha)r^2(x_\beta\delta_{\gamma k}+x_\gamma\delta_{\beta k}) +\alpha(N+2)(N+4)x_\beta x_\gamma x_k+(2\alpha^2-4\alpha-N-2)\delta_{\beta\gamma}x_k]\overline{w}_k dV, \quad (4.13)$$

where $S(R)$ again is required to lie interior to D. An application of Schwarz's inequality then yields

$$|\bar{\tau}_{\beta\gamma}(0)|^2\leq K_{\beta\gamma}(N)R^{-(N+2)}\int_{S(R)}\overline{w}_k\overline{w}_k dV, \quad (4.14)$$

where $K_{\beta\gamma}(N)$ is an easily computable function of N.

To obtain a bound for $\tau'_{\beta\gamma}(0)$ we differentiate (4.7) to obtain

$$\tau'_{\beta\gamma}(0)=\oint_C \tau_{\beta\gamma}(z^k)[w'_{k,j}n_j+\alpha w'_{j,j}n_k]dS+\int_D \tau_{\beta\gamma}(z^k)L_k(w')dV \quad (4.15)$$

where we have used the notation

$$\mu^{-1}\tau_{\beta\gamma}(z^k)=\frac{\partial z^k_\beta(P,Q)}{\partial x_\gamma}+\frac{\partial z^k_\gamma(P,Q)}{\partial x_\beta}+\alpha\frac{\partial z^k_l(P,Q)}{\partial x_l}\delta_{\beta\gamma} \quad (4.16)$$

the differentiation being taken with respect to the variables of the point P. Thus the $\tau_{\beta\gamma}(z^k)$ occurring in (4.15) is formed by first taking $z^k_l(P,Q)$ (the displacement at Q due to a point load at P) performing the indicated differentiation with respect to the variables of the point P and then evaluating at the origin. As before, from Schwarz's inequality we obtain

$$|\tau'_{\beta\gamma}(0)|^2\leq\Big(\Big\{\oint_C t^{-1}[\tau_{\beta\gamma}(z^k)\tau_{\beta\gamma}(z^k)+(\tau_{\beta\gamma}(z^k)n_k)^2]dS \cdot\oint_C t(w'_{k,l}w'_{k,l}+\alpha w'_{k,k}w'_{l,l})dS\Big\}^{\frac{1}{2}}+\Big|\int_D \tau_{\beta\gamma}(z^k)L_k(w)dV\Big|\Big)^2. \quad (4.17)$$

The insertion of (2.19) into (4.17) then leads to the result

$$|\tau'_{\beta\gamma}(0)|^2\leq\Big\{M_{\beta\gamma}(N)\Big[\int_D L_j(w)L_j(w)dV\Big]^{\frac{1}{2}}+\Big|\int_D \tau_{\beta\gamma}(z^k)L_k(w)dV\Big|\Big\}^2 \quad (4.18)$$

where $M_{\beta\gamma}(N)$ is an easily computed constant. The desired bound for $\tau_{\beta\gamma}(0)$ now follows since

$$|\tau_{\beta\gamma}(0)|^2\leq 2\{|\bar{\tau}_{\beta\gamma}(0)|^2+|\tau'_{\beta\gamma}(0)|^2\}. \quad (4.19)$$

5. References

[1] Aquaro, G., Un teorema di media per le equazioni dell'elasticità, Riv. Mat. Univ. Parma **1,** 419–424 (1950).

[2] Bramble, J. H. and Payne, L. E., Bounds for solutions of second order elliptic partial differential equations, Contributions to Differential Equations (in press).

[3] Diaz, J. B. and Greenberg, H. J., Upper and lower bounds for the solution of the first boundary value problem of elasticity. Quart. Appl. Math. **6,** 326–331 (1948).

[4] Diaz, J. B. and Payne, L. E., Mean value theorems in the theory of elasticity. Proc. Third U.S. Congress Appl. Mech., Brown Univ., pp. 293–303 (1958).

[5] Faber, G., Beweis, dass unter aller homogenen Membranen von gleicher Fläche und gleicher Spannung die Kreisförmige den tiefsten Grundton gibt, Sitzber. bayer. Akad. Wiss., pp. 169–172 (1923).

[6] Krahn, E., Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises, Math. Ann. **94,** 97–100 (1924); see also, Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen, Acta. Comm. Univ. Dorp. **A9,** 1–44 (1926).

[7] Love, A. E. H., A treatise on the mathematical theory of elasticity (Dover, New York, 1944).

[8] Payne, L. E. and Weinberger, H. F., New bounds for solutions of second order partial differential equations, Pac. J. Math. **8,** 551–573 (1958).

[9] Prager, W. and Synge, J. L., Approximations in elasticity based on the concept of function space, Quart. Appl. Math. **5,** 241–269 (1947).

[10] Synge, J. L., Upper and lower bounds for the solutions of problems in elasticity, Proc. Roy. Irish Acad. **53,** 41–64 (1950).

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Selected Abstracts

The nonsingular embedding of transition processes with a more general framework of coupled variables, J. Heading, *J. Research NBS* **65D** (*Radio Prop.*), *No. 6,* (*Nov.–Dec. 1961*) *75 cents.*

Reflection and coupling processes exhibited by plane electromagnetic waves propagated in an inhomogeneous horizontally-stratified anisotropic ionosphere are associated with discrete transition points or with continuous coupling regions. These arise when the fourth order differential equations are written in first order coupled form, and many terms in these equations become infinite at the transition points. This procedure is rendered more precise by means of a special linear transformation that reformulates the equations in a new way, thereby exhibiting the manner in which local coupling processes are embedded in the more general background process of otherwise independently propagated characteristic waves. To exhibit the power of the matrix algebra involved, the case of an arbitrary number of characteristic waves is considered; moreover, Försterling-type coupled equations are produced in a more generalized form than hitherto considered, and a discussion of the equations governing continuous coupling completes the paper.

The robustness of life testing procedures derived from the exponential distribution, M. Zelen and M. C. Dannemiller, *Technometrics* **3**, *No. 1, 29–49 (Feb. 1961).*

Almost all the statistical procedures in current use for evaluating the reliability of components or equipment rest on the assumption that the failure times follow the exponential distribution. However, in practical situations one rarely has enough data to determine whether failure times are actually exponential. This paper studies the behavior of several statistical life testing procedures based on the exponential failure law if the true failure law is the Weibull distribution. It is found that these statistical techniques, which are widely used, are very sensitive to departures from initial assumptions. Applying these techniques to life test data when the exponential failure law is not satisfied may result in substantially increasing the probability of accepting components or equipments having poor mean-time-to-failure.

This paper also develops convenient analytic techniques for approximating (i) the distribution of sums of independent random variables, and (ii) the characteristics of sequential procedures, for non-negative random variables. These techniques are based on a general life distribution which consists of an expansion of generalized Laguerre polynomials. Application of these methods to the Weibull distribution yields excellent results.

Applications of graphs and Boolean matrices to computer programming, R. B. Marimont, *SIAM Rev.* **2**, *No. 4, 259–268 (Oct. 1960).*

It is shown how the techniques of graph theory and Boolean matrices may be used in the analysis of digital computer programs for the elimination of inconsistencies and redundancies. The inadequacy of the present methods to deal with the more complicated problems of computer networks is illustrated by discussion of some important unsolved problems.

On some partial differential equations of Brownian motion of a free particle, A. Ghaffar. (*Abstract*) *Proc. Intern. Conf. on Partial Differential Equations and Continuum Mechanics (Univ. of Wisconsin, Madison, Wisc., June 7–15, 1960), 348–350 (1961).*

The most general one-dimensional solutions of the Chapman-Kolmogoroff functional equation are well known as series of parabolic cylinder functions. The present paper gives a derivation of the corresponding parabolic partial differential equations and their simplification and solution for a special case. These solutions differ from those of A. N. Kolmogoroff and are of class $L^{(2)}$ over $(-\infty, +\infty)$. The same method furnishes the most general one-dimensional solutions, as series of Laguerre polynomials which converge uniformly over $(0, +\infty)$ under certain conditions.

Nonequilibrium processes in isotopically disordered crystals. Dependence on degree of order, R. Zwanzig, *J. Math. Phys.*, **2**, *No. 3, 370–372 (Mar.–June 1961).*

A previous paper [A. A. Maradudin, G. H. Weiss, and D. H. Jepsen, J. Math. Phys. 2, 349 (1961)] has shown that approximate normal modes of isotopically disordered crystals decay irreversibly when the arrangement of isotopes is completely random. These results are generalized to crystals with an arbitrary degree of order. In particular, it is shown irreversible behavior occurs whenever the spatial correlation between isotopic species extends over a fixed finite range, as the size of the crystal tends to infinity.

A diffraction theory for LF sky-wave propagation, J. R. Wait, *J. Geophys. Research* **66**, *No. 6, 1713–1724 (June 1961);* **An additional note,** J. R. Wait and A. M. Conda, *J. Geophys. Research* **66**, *No. 6, 1725–1729 (June 1961).*

The concept that radio waves propagate from transmitter to receiver via discrete ionospheric reflections (i.e., hops) is put on a sound theoretical basis. The earth and the ionosphere are represented by two (smooth) concentric spherical surfaces. The (local) reflecting characteristics of these boundaries are assumed to be known. It is shown that the m'th hop sky wave may be generally represented by a complex integral. The geometrical-optical representations for the field are retrieved as a special case. For low-radio-frequencies, it is shown that these optical-type formulae, which are often used, have severe limitations. Generally, they are not valid near the caustic points. Techniques for evaluating the complex integral for the sky waves near and beyond the caustics are then discussed in some detail. Where possible, a physical interpretation of the results is given.

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Journal of Research 65D (Radio Prop.), No. 5 (Sept.–Oct. 1961) 75 cents.

Frequency dependence of *D*-region scattering at VHF. J. C. Blair, R. N. Davis, Jr., and R. C. Kirby.
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On the spectrum of terrestrial radio noise at extremely low frequencies. H. R. Raemer.
The nonsingular embedding of transition processes within a more general framework of coupled variables. J. Heading. (See above abstract.)
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Design of panoramic ionospheric resources. L. H. Heisler and L. D. Wilson.
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Measurements of low-angle radiation from a monopole. A. C. Wilson.

Ideal gas thermodynamic functions and isotope exchange functions for the diatomic hydrides, deuterides, and tritides, L. Haar, A. S. Friedman, and C. W. Beckett, NBS Mono. 20 (1961) $2.75.
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Thermal expansion of technical solids at low temperatures. A compilation from the literature, R. J. Corruccini and J. J. Gniewek, NBS Mono. 29 (1961) 20 cents.
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Publications of the National Bureau of Standards July 1, 1957, to June 30, 1960 (Includes titles of papers published in outside journals 1950 to 1959), B. L. Arnold, NBS Misc. Publ. 240 (1961) $2.25.
Mean electron density variations of the quiet ionosphere—May 3, 1959, J. W. Wright, L. R. Wescott, and D. J. Brown, NBS TN40-3 (PB151399-3) (1961) $1.50.
Bibliography on ionospheric propagation of radio waves (1923–1960), W. Nupen, NBS TN84 (PB161585) (1961) $7.00.
Collisions of liquid drops with liquids, O. G. Engel, NBS TN89 (PB161590) (1961) $1.00.
Determination of the K fluorescence yield of argon by proportional-counter spectrometry, C. Godeau, NBS TN91 (PB161592) (1961) 50 cents.
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Data reduction instrumentation for radio propagation research, W. E. Johnson, NBS TN111 (PB161612) (1961) $1.00.
The nature, cause and effect of the porosity in electrodeposits. VII. A microscopic examination of nickel-chromium coatings after atmospheric corrosion, D. W. Ernst and F. Ogburn, Plating **48,** No. 4, 491–497 (May 1961).
Physical quantities proposed for radiation measurements, L. S. Taylor, Proc. IXth Intern. Congress of Radiology, July 23–30, 1959 (München, Germany), 1301–1302 (1960).
Improving rectifier circuits, G. F. Montgomery, Electronics p. 85–87 (Apr. 7, 1961).
Fracture characteristics of notched tensile specimens of titanium and a titanium alloy, G. W. Geil and N. L. Carwile, Materials Research and Standards (Am. Soc. Testing Materials, Philadelphia, Pa.), **1,** No. 1, 16–21 (Jan. 1961).
Absence of an isotope effect in the fractional recrystallization of alpha-D-glucose-1-*t*, H. S. Isbell, H. L. Frush, and N. B. Holt, Anal. Chem. **33,** 225–226 (Feb. 1961).
Comparison of national standards for roentgen measurement, H. O. Wyckoff, IXth Intern. Congress of Radiology, July 23–30, 1959 (München, Germany), 1315–1318 (1960).
Environmental factors in a family-size underground shelter, R. P. Achenbach, Proc. Meeting on Environmental Engineering in Protective Shelters, Feb. 8–10, 1960, Natl. Acad. Sci.-Natl. Research Council, Washington, D.C., 69–118 (1961).
How to evaluate accuracy, W. J. Youden, Materials Research and Standards (Am. Soc. Testing Materials, Philadelphia, Pa.) **1,** No. 4, 268–271 (April 1961).
Improved standard for the calibration of vibration pickups, R. R. Bouche, Exp. Mech. 1–6 (April 1961).
Absorption and scattering of photons by holmium and erbium, E. G. Fuller and E. Hayward, Proc. Intern. Conf. Nuclear Structure, 760–766 (Kingston, Canada, 1960).
Introductory remarks, U. Fano, Conf. Coherence Properties of Electromagnetic Radiation, AFOSR-583, Report on Conf., 31–38 (University of Rochester, The Institute of Optics, Rochester, N.Y., Apr. 1961).
Available heat sinks for protected underground installations, B. A. Peavy, Proc. Meeting on Environmental Engineering in Protective Shelters, Feb. 8–10, 1960, Natl. Acad. Sci.-Natl. Research Council, Washington, D.C., 69–118 (1961).
Interpretation of the appearance potentials of secondary ions, M. B. Wallenstein and M. Krauss, J. Chem. Phys. **34,** No. 3, 929–936 (Mar. 1961).
Observations on the chemiluminescent reaction of nitric oxide with atomic oxygen, H. P. Broida, H. I. Schiff, and T. M. Sugden, Trans. Faraday Soc. **57,** No. 458, 259–265 (Feb. 1961).
Studies of the low-temperature distillation of hydrogen isotopes, T. M. Flynn, Cryogenics **1,** No. 2, 1–5 (Dec. 1960); Advances Cryogenic Engr. **6,** 236–244 (1961).
Inert enclosed pump for shaped flow of ultraclean solutions, M. W. Barnes and R. H. Noyce, Rev. Sci. Instr. **32,** No. 3, 353 (Mar. 1961).
Radiation protection standards, L. S. Taylor, Radiology **74,** 824–831 (May 1960).

Atomic beam frequency standards, R. C. Mockler, R. E. Beehler, and C. S. Snider, IRE Trans. Instrumentation **I-9,** No. 2, 120–132 (Sept. 1960).

The development of more stable gage blocks, M. R. Meyerson, T. R. Young, and W. R. Ney, ASTM Bull. **1,** No. 5, 368–374 (May 1961).

The power spectrum and its importance in precise frequency measurements, J. A. Barnes and R. C. Mockler, IRE Trans. Instrumentation **I-9,** No. 2, 149–155 (Sept. 1960).

VLF phase perturbation associated with meteor shower ionization, C. J. Chilton, J. Geophys. Research **66,** No. 2, 379–383 (Feb. 1961).

Departures from the Saha equation for ionized helium. I. Condition of detailed balance in the resonance lines, R. N. Thomas and J. B. Zirker, Astrophys. J. **133,** No. 2, 588–595 (Mar. 1961).

Tests for regression coefficients when errors are correlated, M. M. Siddiqui, Annals Math. Stat. **31,** No. 4, 929–938 (Dec. 1960).

Electron diffraction studies on solid α-nitrogen, E. M. Horl and L. Marton, Acta Crystallographica **14,** Pt. I, 11–19 (Jan. 10, 1961).

A comment on the NRL solar Lyman-Alpha results, J. T. Jefferies and R. N. Thomas, Astrophys. J. **133,** No. 2, 606–607 (Mar. 1961).

The nation's electronic standards program: where do we now stand? H. W. Lance, IRE Trans. Instrumentation **I-9,** No. 2, 94–100 (Sept. 1960).

Hard gallium alloys for use as low contact resistance electrodes and for bonding thermocouples into samples, G. C. Harman, Rev. Sci. Instr. **31,** No. 7, 717–720 (July 1960).

Scattering of photons by deformed nuclei, E. G. Fuller and E. Hayward, Proc. Intern. Conf. Nuclear Structure, 763–766 (Kingston, Canada, 1960).

The glass electrode, R. G. Bates, Book, Reference Electrodes, edited by D. J. G. Ives and G. J. Janz (Academic Press Inc., New York, N. Y.), Chapter 5, 231–269 (1961).

A recalibration of the NBS standard thermal neutron flux, E. R. Mosburg, Jr., and W. M. Murphey, J. Nuclear Eng. (Reactor Science and Technology Pt.) **14,** 25–30 (1961).

The relationship of low-height ionosonde echoes to Auroral-zone absorption and VHF *D*-scatter, J. K. Olesen and J. W. Wright, J. Geophys. Research **66,** No. 4, 1127–1134 (Apr. 1961).

Trends in the technology of automatic data processing, S. N. Alexander, AMA Report **43,** 38–42 (1960).

Use of the bulge test for determining mechanical properties of stainless steel foil, D. B. Ballard, Materials Research and Standards **1,** No. 6, 471–473 (June 1961).

Some comments on the use and preparation of hard gallium alloys, G. C. Harman, Rev. Sci. Instr. **32,** No. 4, 465–466 (Apr.1961).

On the nature of equatorial spread, *F*, R. Cohen and K. L. Bowles, J. Geophys. Research **66,** No. 4, 1081–1106 (Apr. 1961).

A study of the chemical reactions in kilning bone char, F. G. Carpenter, and V. R. Deitz, Proc. 6th Tech. Session on Bone Char 1959, p. 321 (Bone Char Research Project Inc., Charlestown, Mass., 1961).

Rate of vaporization of refractory substances, R. F. Walker, J. Efimenkof, and N. L. Lofgren, Book, Physical Chemistry in Aerodynamics and Space Flight (Pergamon Press, Inc., London, England), p. 24–30 (1961).

Analysis of pyrolyzates of polystyrene and Poly(methyl methacrylate) by gas chromatography, F. A. Lehmann and G. M. Brauer, Anal. Chem. **33,** No. 6, 673–676 (May 1961).

A note on phase velocity of VLF radio waves, J. R. Wait and K. P. Spies, J. Geophys. Research **66,** No. 3, 992–993 (Mar. 1961).

Microwave spectrum of isobutylene, Dipole moment, internal barrier, equilibrium conformation, and structure, V. W. Laurie, J. Chem. Phys. **34,** No. 5, 1516–1519 (May 1961).

Intermolecular forces from optical spectra of impurities in molecular crystals, R. W. Zwanzig, Molecular Phys. **3,** No. 4, 305–311 (July 1960).

On the excitation of electromagnetic surface waves on a curved surface, J. R. Wait, IRE Trans. Ant. Prop. **AP-8,** No. 4, 445–448 (July 1960).

Solubility product phenomena in hydroxyapatite-water systems, H. M. Rootare, V. R. Deitz and F. G. Carpenter, Proc. 6th Tech. Session on Bone Char 1959, p. 287 (Bone Char Research Project Inc., Charlestown, Mass., 1961).

Reproducibilities of carbon and germanium thermometers at 4.2°K, M. H. Edlow and H. H. Plumb, Proc. Cryogenic Eng. Conf., Boulder, Colo., Aug. 1960, **6,** 542–547 (1961).

The automatic chloride titrator in the analysis of some ionic constituents, M. D. Peiperl, F. G. Carpenter, and V. R. Deitz, Proc. 6th Tech. Session on Bone Char 1959, p. 53 (Bone Char Research Project Inc., Charlestown, Mass. 1961).

Spin-lattice relaxation in cerous magnesium nitrate, R. P. Hudson and R. S. Kaeser, Il Nuovo Cimento **19,** Serie X, 1275–1277 (Mar. 16, 1961).

The photolysis of ethyl vinyl ether. E. Murad, J. Am. Chem. Soc. **83,** 1327–1330 (1961).

Vapor phase gamma-radiolysis of azomethane, L. J. Stief and P. Ausloos, J. Phys. Chem, **65,** 877–881 (1961).

Electron interferometer studies of iron whiskers, H. A. Fowler, L. Marton, J. A. Simpson, and J. A. Suddeth, J. Appl. Phys. **32,** No. 6, 1153 (June 1961).

Changes in dentures during storage in water and in service, J. B. Woefel, G. C. Paffenbarger, and W. T. Sweeney, J. Am. Dental Assoc. **62,** No. 6, 643–657 (June 1961).

Solar disturbances and radio communication forecasts, J. F. Brockman, Sky and Telescope **XXI,** No. 6, 322–326 (June 1961).

Dissociation constant of the protonated acid form of 2-Amino-2-(hydroxymethyl)-1, 3-propanediol [tris(hydroxymethyl) aminomethane] and related thermodynamic quantities from 0 to 50°, R. G. Bates and H. B. Hetzer, J. Phys. Chem. **65,** 667–671 (1961).

Hyperfine structure and isotope shifts in the 2537-A line of mercury, W. G. Schweitzer, Jr., J. Opt. Soc. Am. **51,** No. 6, 692–693 (June 1961).

Long-distance one-hop F_1 propagation through the auroral zone, L. H. Tveten, J. Geophys. Research **66,** No. 6, 1683–1684 (June 1961).

Resonators for millimeter and submillimeter wavelengths, W. Culshaw, IRE Trans. Microwave Theory and Tech., **MTT-9,** No. 2, 135–144 (March 1961).

Paramagnetic substances for nuclear orientation, R. P. Hudson, Prog. in Cryogenics, **3,** 99–127 (1961).

A five-attribute system of describing visual appearance, D. B. Judd, Am. Soc. Testing Materials Spec. Tech. Pub. No. 297 (1961).

Circumferentially uniform electroplating of tube bores, J. P. Young and H. I. Salmon, Plating **48,** No. 7, 783; 787–8 (July 1961).

Solar flare effects in the *F* region of the ionosphere, R. W. Knecht and K. Davies, Nature **190,** No. 4778, 797–798 (May 1961).

Age to indium resonance for D-D neutrons in heavy water, V. Spiegel, Jr., and A. C. B. Richardson, Nuclear Sci. and Eng. **10,** No. 1, 11–15 (1961).

Magnetic properties and optical absorption spectrum of K_2ReCl_6, J. C. Eisenstein, J. Chem. Phys. **34,** No. 5, 1628–1648 (May 1961).

Convective circulation in water induced by evaporation cooling, W. G. Spangenberg and W. R. Rowland, Phys. of Fluids, **4,** No. 6, 743–750 (June 1961).

Photoionization heating in the *F* region of the atmosphere, D. C. Hunt and T. E. Van Zandt, J. Geophys. Research **66,** No. 6, 1673–1682 (June 1961).

Identification of fiber blends by infrared spectroscopy, M. K. Wharton and F. H. Forziati, Am. Dyestuff Rep. **50,** No. 14, 33–36 (July 1961).

Residual losses in a guard-ring micrometer-electrode holder for solid-disk dielectric specimens, A. H. Scott and W. P. Harris, 1960 Annual Report, Conference on Electrical Insulation, Natl. Acad. Sci.-Natl. Research Council, 11–17 (1961).

Calculations of the neutron age in water and heavy water for D-D sources, J. W. Cooper, Nuclear Sci. and Eng. **10,** No. 1, 1–10 (May 1961).

Magnetic susceptibility of cerous magnesium nitrate, R. P. Hudson and W. R. Hosler, Phys. Rev. **122,** No. 5, 1417–1420 (June 1961).

Use of gaussian orbitals for atoms-in-molecule calculations, M. Kraus, J. Chem. Phys. **34,** No. 2, 692–693 (February 1961).

Measurement of minority carrier lifetime in SiC by a novel electroluminescent method, G. C. Harman and R. L. Raybold, J. Appl. Phys. **32,** No. 6, 1168–1169 (June 1961).

Absolute measurement of W for Po^{210} alpha particles in air, nitrogen, and carbon dioxide, Z. Bay, P. A. Newman, and H. H. Seliger, Radiation Research **14,** No. 5, 551–565 (May 1961).

Comparative *p*H measurements on papers by water extraction and glass electrode spot tests, J. H. Flynn and L. E. Smith, Tappi **44,** No. 3, 223–228 (Mar. 1961).

The structure of the monoclinic form of sodium tetrametaphosphate tetrahydrate, H. M. Ondik, C. H. MacGillavry, and S. Block, Acta. Crystallographica **14,** 555 (1961).

A comparison between theoretical and experimental data on phase velocity of VLF radio waves, J. R. Wait, Proc, IRE **49,** No. 6, 1089–1090 (June 1961).

Mass spectrometric study of cyanogen and cyanoacetylenes, V. H. Dibeler, R. M. Reese, and J. L. Franklin, J. Am. Chem. Soc. **83,** 1813–1818 (1961).

An analysis of errors in the calibration of electric instruments, F. L. Hermach, Communication and Electronics, 1–6 (May 1961).

Energy levels and magnetic dipole transitions in the $4p^4$ ground configuration of singly ionized atomic bromine (Br II), W. C. Martin and J. L. Tech, J. Opt. Soc. Am. **51,** No. 6, 591–594 (June 1961).

An improved structure determination for vinyl fluoride, D. R. Lide, Jr., and D. Christensen, Spectrochim. Acta **17,** 665–668 (1961).

Incoherent scattering by free electrons as a technique for studying the ionosphere and exosphere: Some observations and theoretical considerations, K. L. Bowles, pp. 223–243 of AGARDograph **42,** The upper atmosphere above *F2*-maximum. North Atlantic Treaty Organization, Advisory Group for Aeronautical Research and Development, Avionics Panel. Papers presented at the Symposium of the Ionospheric Research Committee, Paris, France, May 1959.

A *p*H standard for blood and other physiologic media, V. E. Bower, M. Paabo, and R. G. Bates, Clin. Chem. **7,** No. 3, 292 (June 1961).

Evidence on the laminar nature of the exosphere obtained by means of guided high-frequency wave propagation, R. M. Galley and W. F. Utlaut, Phys. Rev. Letters **6,** No. 11, 591–595 (June 1, 1961).

Comparison of the ionization produced in air by alpha particles near 5 mev and by beta particles, Z. Bay and P. A. Newman, Radiation Research **14,** No. 5, 566–572 (May 1961).

A century-old razor, S. J. Rosenberg, Metal Progress, 94–96 (1961).

Some applications of Hardy's formula in Brownian motion (Abstract), A. Ghaffari, Am. Math. Soc. Notices **8,** No. 54, 238 (June 1961).

On the classical approximation in the statistical theory of mass spectra, H. M. Rosenstock, J. Chem. Phys. **34,** 2182 (1961).

Optical calibration of vibration pickups at small amplitudes, V. A. Schmidt, S. Edelman, E. R. Smith, and E. Jones, J. Acoust. Soc. Am. **33,** No. 6, 748–751 (June 1961).

Superconductivity of Nb_3Sn in pulsed fields of 185 kilogauss, V. D. Arp, R. H. Kropschot, and J. H. Wilson, Phys. Rev. Letters **6,** No. 9, 452–453 (May 1961).

Proposed nomenclature for linear viscoelastic behavior, H. Leaderman and F. Schwarzl, Rheologica Acta, **1,** No. 4/6, 674–676 (1961).

FM and SSB radiotelephone tests on a VHF ionospheric scatter link during multipath conditions, IRE Trans. Commun. Systems **CS–8,** No. 3, 813–186 (Sept. 1960).

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